



# **ALAGAPPA UNIVERSITY**

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**KARAIKUDI – 630 003**



## **Directorate of Distance Education**

### **Master of Business Administration**

**I - Semester**

**317 14**

# **QUANTITATIVE TECHNIQUES**

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# SYLLABI-BOOK MAPPING TABLE

## Quantitative Techniques

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## INTRODUCTION

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### NOTES

Mathematics is the most precise language known to humans. The only way to make our decision-making process precise is to quantify the information we have and then perform calculations on it to arrive at desired solutions. Quantitative methods make all this possible for us. These methods help us assimilate information in a numerical format and perform operations on it to know the outcome, which in turn helps us in decision making.

Quantitative techniques or methods are of paramount importance in the business world, where our success entirely depends on our ability to take correct and timely decisions. Today, most of the business problems have found their quantitative representation, which makes this field of study more interesting and useful to the students and businesspersons alike. These methods are a great aid to the business strategists. In the present world, knowledge of these methods is a prerequisite to enter any business environment.

This book, *Quantitative Techniques*, is divided into fourteen units that follow the self-instruction mode with each unit beginning with an Introduction to the unit, followed by an outline of the Objectives. The detailed content is then presented in a simple but structured manner interspersed with Check Your Progress Questions to test the student's understanding of the topic. A Summary along with a list of Key Words and a set of Self-Assessment Questions and Exercises is also provided at the end of each unit for recapitulation.



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**BLOCK - I**  
**BASICS OF QUANTITATIVE TECHNIQUES**

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**NOTES**

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**UNIT 1 BASIC QUANTITATIVE  
CONCEPTS**

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**Structure**

- 1.0 Introduction
- 1.1 Objectives
- 1.2 Place of Quantitative Analysis in the Practice of Management
- 1.3 Problem Definition: Models and their Development
- 1.4 Concept of Trade-off /Opportunity Cost
- 1.5 Notion of Constants and Variables
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**1.0 INTRODUCTION**

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Quantitative techniques are the powerful tools through which production can be augmented, profits maximized, costs minimized and production methods can be oriented for the accomplishment of certain predetermined objectives. The study of quantitative techniques is a relatively new discipline which has its wide range of applications specially in the field of agriculture and industry. Of late, there has been a growing tendency to turn to quantitative techniques as a means for solving many of the problems that arise in a business or industrial enterprise. A large number of business problems, in the relatively recent past, have been given a quantitative representation with considerable degree of success.

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**1.1 OBJECTIVES**

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After going through this unit, you will be able to:

- Highlight the importance of quantitative analysis in management practices
- Explain the reasons for using models and the methods of developing them
- List the different types of constants

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## 1.2 PLACE OF QUANTITATIVE ANALYSIS IN THE PRACTICE OF MANAGEMENT

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### NOTES

Quantitative techniques are those statistical and operations research or programming techniques which help in the decision-making process specially concerning business and industry. These techniques involve the introduction of the element of quantities, i.e., they involve the use of numbers, symbols and other mathematical expressions. The quantitative techniques are essentially a helpful supplement to judgement and intuition. These techniques evaluate planning factors and alternatives as and when they arise rather than prescribe courses of action. As such, quantitative techniques may be defined as those techniques which provide the decision maker with a systematic and powerful means of analysis and help, based on quantitative data, in exploring policies for achieving pre-determined goals. These techniques are particularly relevant to problems of complex business enterprises.

Numerous quantitative techniques are available in modern times. They can broadly be put under two groups: (a) Statistical techniques (b) Programming techniques.

*Statistical techniques* are those techniques which are used in conducting the statistical inquiry concerning a certain phenomenon. They include all the statistical methods beginning from the collection of data till the task of interpretation of the collected data. More clearly, the methods of collection of statistical data, the technique of classification and tabulation of the collected data, the calculation of various statistical measures such as mean, standard deviation, coefficient of correlation, etc., the techniques of analysis and interpretation and finally the task of deriving inferences and judging their reliability are some of the important statistical techniques.

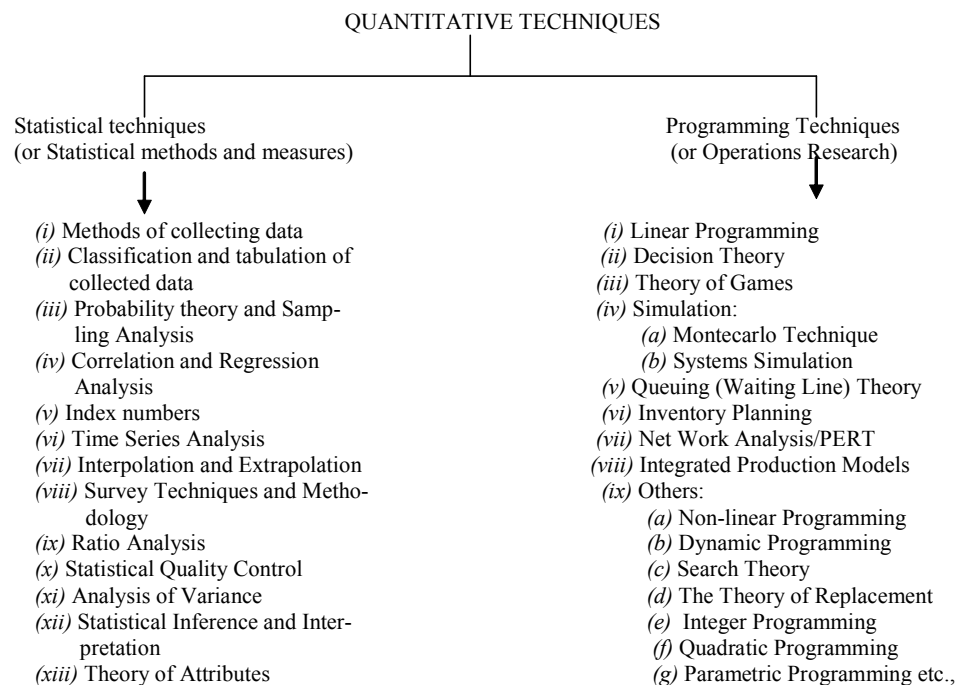
*Programming techniques* (or what is generally described as Operations Research or simply OR) are the model building techniques used by decision makers in modern times. They include wide variety of techniques such as linear programming, theory of games, simulation, network analysis, queuing theory and many other similar techniques. The following steps are generally involved in the application of the programming techniques:

- (1) All quantifiable factors which are pertinent to the functioning of the business system under consideration are defined in mathematical language: variables (factors which are controllable) and parameters or coefficients (factors which are not controllable).
- (2) Appropriate mathematical expressions are formulated which describe inter-relations of all variables and parameters. This is what is known as the formulation of the mathematical model. This model describes the technology and the economics of a business through a set of simultaneous equations and inequalities.

- (3) An optimum solution is determined on the basis of the various equations of the model satisfying the limitations and interrelations of the business system and at the same time maximizing profits or minimizing costs or coming as close as possible to some other goal or criterion.
- (4) The solution values of the model, obtained as above, are then tested against actual observations. If necessary, the model is modified in the light of such observations and the whole process is repeated till a satisfactory model is attained.
- (5) Finally, the solution is put to work.

From the above it becomes clear that programming techniques involve the building up of a mathematical model, i.e., sets of equations and inequalities relating significant variables in a situation to the outcome. These techniques give solution to problems in terms of the values of the variables involved. These techniques assist specially in resolving complex problems. These techniques, developed in the 1940's first to tackle defence and military problems, are now being applied increasingly to business situations.

The following chart enlists the names of the important quantitative techniques:



A brief mention of the meaning of various important quantitative techniques will be appropriate at this juncture. This is as under:

### (1) Statistical Techniques

Some of the important statistical techniques often used in business and industry are:

- (i) *Probability theory and sampling analysis.* In many studies concerning business problems, the considerations of time and cost lead to examinations

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of only a few items of the universe. But the items selected should be as representative as possible of the total population. The selection process is called sampling. Samples are of two basic types:

- (a) *Probability or random samples* are those which are so constructed that every element or item from the total population has equal probability of selection and the limits of probable error in relating results to the whole population are known mathematically in advance.
- (b) *Non-probability or purposive samples* are those which are based on the choice of the selector.

The sampling analysis through the use of various designs makes possible to derive inferences about population characteristics with specified degree of reliability. This statistical technique is often used in market research, work sampling, inventory control and auditing.

- (ii) *Correlation and regression analysis:* Correlation and regression analysis is another important statistical technique often used in business and industry. ‘Regression’ analysis examines the past trends of relationships between one variable, e.g., sales volume, and one or more than one other variables, e.g., advertising expenditure, cost of salesmen. Correlation analysis measures the closeness of such relationships. Thus, the ‘correlation’ and ‘regression’ analysis is used to study the degree of functional relationship among two or more variables and through this technique the value of one variable can be estimated if the value of another variable is known.
- (iii) *Index numbers:* ‘Index Numbers’ constitute that statistical technique which measures fluctuations in prices, volume, economic activity or other variables over a period of time, relative to a base. The choice of the base period, the method of weighting and the selection of the components to be included in the index are the key factors concerning this statistical technique. Index numbers play an increasingly important role these days in decision-making problems for private enterprises as well as for the Government.
- (iv) *Time series analysis:* Through this statistical technique, series of data over a period of time are analysed as to their chief types of fluctuations such as trend, cyclical, seasonal and irregular. Such an analysis helps the management in the field of interpreting sales, production, price or other variables over a period of time. This technique is of considerable significance in the field of short and long term business forecasting and greatly facilitates the seasonal as well as future planning of business operations.
- (v) *Interpolation and extrapolation:* Interpolation is the statistical technique of estimating, under certain assumptions, the figures missing amongst the given values of the variable itself whereas extrapolation provides figures outside the given data. According to *W.M. Harper* — Interpolation consists in reading a value which lies between two extreme points; extrapolation means reading a value that lies outside the two extreme points.” This technique

helps to ascertain the probable prices, business changes, the probable production and the like. It also fills the gaps in the available data besides being of great help in business forecasting.

- (vi) *Ratio analysis*: Ratio analysis technique as applied to business is a part of whole process of analysis of financial statements of any business or industrial concern to take credit decisions. This technique has emerged as most useful to bankers to study their customers' balance sheets so as to have the clear idea about the growth of the concerns and also about the future trend of progress of such concerns. Under this technique various ratios (a ratio is simply one number expressed in terms of another) are worked out and interpreted with a view to find out mainly the financial stability, liquidity, profitability and the quality of management of the business and industrial concerns.
- (vii) *Statistical quality control*: The technique of statistical quality control is used by almost all the modern manufacturing industries. Under this technique the control of the quality is ensured by the application of the theory of probability to the results of examination of samples. The technique helps in separating the assignable causes from the chance causes. The technique of statistical quality control is applied through two phases—'Process Control' and 'Acceptance Sampling.'
- (a) *Process control* (sometimes also known as control chart technique) is the application of statistical tools to industry to maintain quality of products. The standard of products are specified to which the quality must conform and then through the use of various control charts (X-Chart, R-Chart etc.), it is seen whether the process is under control. The object of all this is to control the quality during the process of manufacture and thus to ensure that the quality of manufacturing is satisfactory and according to specific standards. Through this technique unnecessary waste of materials, time, etc., is avoided first by detecting faulty production and its causes and then by taking the corrective action immediately. This is how the quality of the product is ensured under statistical quality control.
- (b) *Acceptance sampling* (also known as product control lot acceptance sampling plan) is that phase of statistical quality control which attempts to decide whether to accept a lot with a desirable quality level or to reject a lot with an undesirable quality level on the basis of evidence provided by inspection of samples drawn at random from the lot. Thus the object of the acceptance sampling is to accept or reject a lot, once it has been manufactured. This helps in taking appropriate decisions concerning the purchase of a given lot.
- (viii) *Other statistical techniques*: Besides the statistical techniques mentioned above, there are other statistical techniques which can as well be used in different situations. One important technique is that of the *analysis of*

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*variance*. This technique is essentially, ‘a method of analysing the variance to which a response is subject, into its various components corresponding to the sources of variation which can be identified.’ In simple words, the analysis of variance is a method of splitting the total variation of the given data into constituent parts which measure different sources of variation. This technique is used to test for the equality of the several sample means, usually more than two. For instance, three varieties of wheat are planted on several plots and their yields per hectare recorded. We might be interested in testing the null hypothesis that the three varieties produce an equal yield on the average. This can easily be done by applying the technique of the analysis of variance. This technique is of great significance in all the research studies concerning phenomena which are capable of quantitative measurements for ‘testing the differences between different groups of data for homogeneity.’ But when the phenomena cannot be measured quantitatively and we are interested in knowing the relationship (technically called the association) between two or more of such phenomena, we can use the technique of what is known as the *theory of attributes*. Under this technique, the coefficient of association and the like measures can be worked out and the inferences about the association between attributes can be drawn. With the help of such inferences, policy decisions can be taken.

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### 1.3 PROBLEM DEFINITION: MODELS AND THEIR DEVELOPMENT

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In general terms, a problem exists when there is a situation that presents doubt, perplexity or difficulty; or when a question is presented for consideration, discussion or solution. The starting point of a problem definition is the information collected during the problem analysis stage. The different aspects related to the design problem must be analysed and should be taken into account in the problem definition. A clearly written and specified problem definition provides a shared understanding of the problem and its relevant aspects.

Operational research and quantitative techniques can be considered as being the application of scientific method by inter-disciplinary teams to problems involving the control of organized (man-machine) systems so as to provide solutions which best serve the purposes of the organization as a whole.

“OR is the application of scientific methods, techniques and tools to problems involving operations with optimum solutions to the problems.”

Churchman, Ackoff and Arnoff

Churchman, Ackoff and Arnoff defined that operations research problems can be solved by applying scientific methods, techniques and tools to obtain optimum solutions. Thus, operations research is a systematic and analytical approach to decision-making and problem solving.

“Operations research is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control”.

P.M. Morse and G.F. Kimball

P.M. Morse and G.F. Kimball defined that operations research uses the scientific method which includes mathematical models and statistical techniques for making effective decisions within the state of the current information to reach a proper decision. The purpose of OR is to help management determine its policies and actions scientifically.

The scientific method is a logical, formalized and systematic observation of the phenomenon in order to establish a behaviour of the system containing the problem, formulation of the hypothesis describing the interaction of the factors involved, verification of this hypothesis and implementation of the results. These steps are as follows:

**Step 1: Observation.** After the problem has been identified and isolated, all relevant information about the problem is gathered and objectively recorded and refined. This information helps in defining the problem more accurately so that the entire system can be observed and understood.

**Step 2: Formulation of hypothesis.** A hypothesis is what the manager believes will occur, based upon the information so gathered. Based upon this belief, a course of action can be established. The belief or the hypothesis should be such, that if a certain course of action is adopted, then the outcome will be reasonably predictable. However, the data gathered must support the hypothesis, if the hypothesis is to be accepted for further action.

**Step 3: Testing and verification of hypothesis.** In this stage, a course of action is adopted and the investigator verifies the correctness of the hypothesis by observing the results of the decision. For example, the problem for a college may be the decline in student enrolment. A hypothesis may be that offering courses over the weekend will improve enrolment. This hypothesis or belief may be based upon the information that most people have jobs and cannot attend college during the daytime. This hypothesis is tested by introducing some weekend courses. If successful, the hypothesis is accepted and further decisions are based on this hypothesis. If the weekend programme is unsuccessful, then this particular hypothesis is rejected and another hypothesis formulated and other alternative courses of action are tested.

**Step 4: Implementation, monitoring, and control.** When a course of action is accepted and is successful, the decision is implemented. However, the action is not considered complete until the effects of the decision are closely monitored and the results continue to coincide with expectations. In addition, an effective system of control is needed to detect any deviations from the norm and if possible, an indication of any corrective action that may be needed. This control process should be imbedded in the system so that continuous feedback is automatically measured

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and if there are any changes in any aspects or parameters of the system, then these can be quickly detected and rectified.

*Models:* A model is the most effective and most often used tool of management science for solving intense and complex problems. A model is simply a representation of a system or an object or a realistic situation in some form, formulated to explain the behaviour of the entity itself. It simplifies the real life situation which it reflects. For example, an organization chart is a form of a model representing the structure of the organization. A model is constructed in such a manner that it approximates those aspects of reality that are being investigated so that the system can be analysed and better decisions can be made.

*The Systems Orientation in Developing a Model:* The systems approach promotes the idea that the organization is an open system where the parts and all the sub-systems interact with each other as well as with the external environment. Each part and subsystem should be examined from the point of view of the entire and overall system of the organization. This means that the management must understand the overall effect upon the entire organization if there are any changes made in its components or subsystems. This means that the sub-optimization of anyone functional area cannot be isolated from its effects on other functional areas and the entire system. In other words, suboptimization in individual areas should also lead to optimization of the system. For example, a cost reduction programme in the area of personnel, if conducted in a participative decision-making manner, should not adversely affect any functional or behavioural areas including the employees morale and job satisfaction, but instead should improve the overall organizational situation.

*The Inter-disciplinary Approach in Developing a Model:* The discipline of management science is primarily used to solve highly complex problems with a multitude of inter-related variables, with multidimensional aspects, so that views of various specialists from diversified backgrounds must be integrated so that this diversity produces a synergistic effect. The inter-disciplinary approach becomes highly useful for many management problems which have physical, psychological, biological, sociological, technical and economic aspects. When specialists from various disciplines are called upon, some new and advanced approaches to old problems are often obtained.

### **Reasons for Using Models**

Some of the logical reasons for using the models are:

1. *The model can yield information.* Relative to the real life situation that a model presents, it does so at a much lower cost. Knowledge can be obtained more quickly and for conditions that may not be observable in real life.
2. *Complexity.* Sometimes, there are so many diversified variables involved in a given problem, that it makes the problem highly complex. Accordingly, the management must limit the dimensions of the problem so that it becomes



a comprehensive one and is represented in some form of a model where the behaviour of the variables can be easily studied and manipulated.

3. *Experimentation.* There are many situations in which it becomes desirable to experiment and test prototype models of certain products, before they are marketed. For example, a model airplane can be built and studied for any problems or modifications. A model of a new car or a new computer is rigorously tested in as many real life conditions as possible before putting it in the market.
4. *Orientation with the future.* Modelling is the only systematic way for visualizing patterns in the future. Since the future phenomenon cannot be observed in the present, it must be predicted, based upon the potential outcomes of alternative solutions. Thus, models have been built to forecast the future event related to GNP, economic conditions, rate of employment, inflation, population movements etc. Such knowledge about the a future events is important for the survival of any organization. Management science models are very powerful tools in predicting the future effects of various decisions.

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### Types of Models

There are three basic and general categories of models. These are:

- (a) *Iconic or physical models.* Iconic model is the least abstract in nature and is a physical replica of the entity in a scale down or scale up design and is in a sense a look alike of the object it represents. A three-dimensional model of a ship or an airplane is an example of conic modelling. Similarly, blue prints of a plant, small scale model of a building or a picture or photograph of an item are other examples.
- (b) *Analog models.* The analog model is a graphical or physical representation of a real object or a situation, but without the same physical appearance as the object or situation itself. The model does not look like the real system but behaves like it. It may also be a diagrammatical representation of relationships of interdependent variables. For example, a graph depicting the relationship between the cost per unit produced and the volume in units produced would be an analogue model of the real relationship.  
Similarly, an organization chart represents the chains of command as well as the formal relationships between individuals and activities. The differing colours on a map may represent oceans or continents. A state or a country map is an analogue model showing roads, highways and towns, and their distances. Similarly, a thermometer is an analogy for temperature and a speedometer represents speed.
- (c) *Symbolic or mathematical models.* These models are used to describe actual problems. They may be verbal in nature like a report describing a problem or a book explaining a situation and events. However, the most

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commonly used models are the mathematical models which are of prime interest to the decision maker. These models use mathematical symbols and relationships to describe the properties or characteristics of a system, object, event or reality. These models are most abstract in nature and can be manipulated by using some known laws of mathematics. Operations research techniques used in various disciplines are primarily based upon mathematical modelling. For example,  $y = a + bx$  is a symbolic model for any straight line where  $a$  and  $b$  are respectively the intercept on  $y$ -axis at value  $x = 0$ , and the slope of the line. The most famous example is Einstein's equation  $E = mc^2$ , which describes relationship between matter and energy. This relationship became the foundation for nuclear and quantum physics.

### Some Other Classification of Models

In addition to the above three basic categories, some models can be classified on the basis of factors such as the purpose of the model and methodologies used. Some of these classifications are:

1. *Normative versus descriptive models.* A normative model is designed to prescribe what ought to be done. It is a prescription of rules for accomplishing some specific objectives in an optimum manner. The descriptive models, on the other hand, describe facts and relationships. These are designed to present occurrences and situations as they are and do not provide specific solutions to problems. However, they are oriented towards answering 'what if' type of questions and may identify effects if some of the problem variables are changed.
2. *Deterministic versus probabilistic models.* Deterministic models assume the conditions of perfect and accurate knowledge and the variables can be precisely quantified. Since these variables are constant and known, it implies that the results and outcomes of each decision and strategy are precisely known. Linear programming, assignment models and transportation models are some examples of this type.

The probabilistic models, on the other hand, reflect situations in which the outcomes of the decisions are not unique and cannot be predicted with total certainty. The behaviour of certain variables in the model is subject to chance and hence any decision taken can result in more than one outcome. For example, the consumer response to advertising of a product cannot be accurately measured and hence advertising decisions would generally be probabilistic in nature.

3. *Static versus dynamic models.* The static decision model is a constant model where the given values of the variables do not change over a given period of time. It does not consider the impact of change that might take place in the prescribed period of time. The linear programming model, for example, is a static model.

The dynamic models, on the other hand, make provisions for the time element to be considered, as one of the variables so that as the time passes, the model can be modified to include the effect of any changes in any variables that might have taken place over the predetermined time period. An example would be monthly production schedules when the decisions would correspond to any changes occurring during the month.

4. *Linear versus non-linear models.* Linear models are characterized by the premise that all components of the model have a linear behaviour and relationship. This relationship is identified by the factors of proportionality and additivity. Proportionality, for example, means that profits are directly proportionate to production and sales, meaning that doubling the production will double the profits as well. The additivity factor means that the effect of two joint programs is the same as the sum of their individual programs.

On the other hand, a non-linear model is identified by the premise that one or more components of the model have a non-linear relationship.

5. *Empirical versus theoretical models.* These models are identified by the differences in methodology that is applied to them. In empirical models, the probabilities that are assigned to the likelihood of events are based upon, or calculated from the past experiences of similar situations. In theoretical models, these probabilities are assigned on the basis of mathematical calculations which in turn are based upon some accepted assumptions about the mechanisms that generate the outcome of events. For example, the probability of a head in the toss of a coin is  $1/2$ . This calculation is based upon the assumption that there are only two possible outcomes of the tossing of a fair coin and each outcome is equally likely to occur.
6. *Simulation versus heuristic models.* The simulation models have been developed where the computers are extensively used. These models are general replica of the real system and are in the form of a computer program. It generally consists of a mathematical structure but is not solved mathematically to yield a general solution. But instead the model is experimented upon by inserting into the model, specific values of decision variables which would yield a series of output. Simulation results are inferential in nature and may not be very precise. For example, we can test the effects of different number of teller positions in a bank under given condition of customers and their rate of arrival at the bank. A simulation imitates a real system so that we can study the properties and behaviour of the real life system.

Heuristic models employ some general intuitive rules or some rule of thumb and are not designed to yield optimal solutions. They just develop good enough or simply workable or satisfactory solutions within the cost or budgetary constraints. According to Simon, most people, whether they are involved in individual decision-making or organizational decision-making, are willing to accept a satisfactory

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alternative which may be something less than the best. The decision maker usually sets up a desired level of goals and then searches for alternatives until one is found that achieves this.

For example, serving customers in a bank on a 'first come, first served' basis may not be the best solution but a satisfactory solution. According to Weist, Heuristic programming techniques have been applied to the areas of assembly line balancing, facilities layout, warehouse location, department store pricing, etc.

### The Model Building Process

The following are the basic steps of the model building process:

1. *Problem formulation.* The first and the most critical step in building a model is the accurate and comprehensive formulation of the problem. This formulation must include a precise description of goals and objectives to be achieved and identification of decision variables, both controllable as well as uncontrollable and any constraints or restrictions on the variables or the outcomes. In addition, the symptoms must be differentiated from causes.
2. *Construction of the model.* Out of a number of methodologies and models available, the management scientist must determine the most suitable model that would represent the system, meet the objectives and yield the desired output. Other factors to be considered are cost and human response. A model that costs more than the problem it intends to solve is not justified. Similarly, a model should be accepted by the people involved in it, those that are responsible for implementing the results of the model and those that are affected by it as a necessary element, rather than a threat to them.
3. *Solution of the model.* If the model fits into some previously established and well-known mathematical models such as linear programming, then the optimal solutions can be more easily obtained. However, if some of the parameters and mathematical relationships are too complex and do not conform to analytical solutions, then simulation techniques may be used. These techniques may not give us the optimal solutions but only feasible, workable and good solutions.
4. *Testing the model.* The testing of the model would determine whether all relevant components of the real situation are incorporated in the model. An effective model should be able to predict the actual system's performance. One way to ascertain its validity is to compare its performance with past performance of the actual system.
5. *Implementation.* Before the model is implemented, the managers must be trained to understand and use the model including knowledge about the functioning of the model and its potential uses and limitations. In addition, the model must be updated whenever necessary if there are any changes either in the organizational objectives or in the external environment. The

changes in the external environment may consist of new customers, new suppliers or new technology that might invalidate the assumptions and information upon which the model is based.

### Some Problems with Modelling

Like all other techniques and methodologies, management science models also need to be completely understood, i.e., if these have to be applied successfully. The effectiveness of these models is reduced due to certain potential problems. Some common problems are:

1. *Invalid assumptions.* These models are only effective when the underlying assumptions and the variables used in the model are true reflection of the real world environment. Inaccurate assumptions will lead to inaccurate solutions. For example, a model cannot be used to project inventory requirements unless sales projections of the coming period are accurate.
2. *Information limitations.* It is not always possible to obtain correct information on/all relevant factors. Incorrect information leads to invalid assumptions. Some information may not be available at all. Some other may not be measurable. It may be difficult to separate facts from inferences and assumptions. If the environment is volatile and excessively dynamic, the information may need to be updated quickly which may not be possible or feasible.
3. *Lack of understanding between the management and management scientists.* Line managers are typically action oriented and usually lack the understanding of sophisticated and complex mathematical models and their effectiveness. The scientific analyst is more interested in optimal solutions. Hence, the operational managers must be properly trained in these areas by the quantitative staff specialists with continuously open lines of communication.
4. *Resistance to change.* Poor implementation of the model is partly due to fear of using the model and partly due to natural resistance to change. Line managers usually have their own ways of solving operational problems and may resist new methods which are highly involved and technically complex. According to Kearney and Martin, "Many managers would rather live with a problem that they cannot solve, than use a solution they do not understand." Accordingly, it will be helpful if the managers are encouraged to participate in the entire process of model building.
5. *Cost.* The model's benefits must outweigh its costs. More sophisticated techniques require considerable financial investment in the form of specialized staff, time, computer capability, the cost of information gathering and processing, training of line managers, etc. Accordingly, this cost factor often discourages the widespread use of quantitative technique, even with the fast evaluation of alternatives generated by high speed computers.

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## 1.4 CONCEPT OF TRADE-OFF /OPPORTUNITY COST

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A trade-off (or tradeoff) is a situation that involves losing one quality or aspect of something in return for gaining another quality or aspect. It implies a decision to be made with full comprehension of both the upside and downside of a particular choice.

In economics the term is expressed as opportunity cost, referring to the most preferred alternative given up. A trade-off, then, involves a sacrifice that must be made to obtain a certain product, rather than other products that can be made using the same required resources. For a person going to a basketball game, its opportunity cost is the money and time expended, say that would have been spent watching a particular television programme.

Opportunity cost is the sacrifice involved in accepting an alternative under consideration. In other words, it is a cost that measures the benefit that is lost or sacrificed when the choice of one course of action requires that other alternative course of action be given up. For example, a company has deposited ₹1 lakh in bank at 10 per cent p.a. interest. Now, it is considering a proposal to invest this amount in debentures where the yield is 17 per cent p.a. If the company decides to invest in debentures, it will have to forego bank interest of ₹10,000 p.a., which is the opportunity cost. Opportunity cost is a pure decision-making cost. It is an imputed cost that does not require a cash outlay and it is not entered in the accounting books.

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## 1.5 NOTION OF CONSTANTS AND VARIABLES

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Let us understand the notion of constants and variables.

### 1.5.1 Constants

A symbol which retains the same value throughout a particular problem is called a **constant**. Constants are usually denoted by letters at the beginning of the alphabet:  $a, b, c$ , etc. Constants may be absolute or arbitrary.

*Absolute Constants* : An absolute or numerical constant retains the same value in all problems and operations.  $2, -5, 7, \pi, e$  are absolute constants.

*Parameters* : An arbitrary or parametric constant is a symbol which is constant for the purpose of any particular problem but may assume different values in different problems. The value of a parameter depends on the particular situation represented in the problem.

### 1.5.2 Variables: Notion of Mathematical Models

A symbol which assumes different values in a given problem is called a **variable**. A variable is usually denoted by the last letters of the alphabet:  $x, y, z$ , etc. The set of values which a variable assumes is its *range*. A variable may be continuous or discrete.

A variable is any characteristic which can assume different values. Age, height, IQ and so on are all variables since their values can change when applied to different people. For example, Mr. X is a variable since X can represent anybody. On the other hand, a *constant* will always have the same value. For example, the number of days in a week are constant and will always remain the same. Consider the following illustration:

Let,  $x + 6 > 10$  be an inequality. Now, if  $x$  is a whole number, then it can have any value greater than 4. While the values 6 and 10 are constant and do not change,  $x$  can be 5, 6, 7... and up to any value. Thus,  $x$  is a variable which can have any number of different values.

There are two types of variables. One type is known as *discrete variable* and the other is known as *continuous variable*. A discrete variable takes whole number values and consists of distinct, recognizable individual elements that can be counted, such as the number of books in a library. Similarly, number of children in a family would be considered as values of a discrete variable, since the children can be counted exactly.

On the other hand, a continuous variable is a variable whose values can theoretically take on an infinite number of values within a given range of values.

Hence, these values are measured as against being counted. However, since the measurement value would depend upon how accurately we measure it, any exact value, would simply be one of the infinite number of values on a continuous scale between two given points. For example, the height of a child touches every one of the infinite number of points between, let us say, 40 inches and 40.1 inches as he/she grows from 40 inches to 40.1 inches. Accordingly, the value of a continuous variable is more accurately defined if it is stated as being between two points such as 40 inches and 40.1 inches.

### **A Random Variable**

A random variable is a phenomenon of interest in which the observed outcomes of an activity are entirely by chance, are absolutely unpredictable and may differ from response to response. By definition of randomness, each possible entity has the same chance of being considered. For instance, lottery drawings are considered to be random drawings so that each number has exactly the same chance of being picked up. Similarly, the value of the outcome of a toss of a fair coin is random, since a head or a tail has the same chance of occurring.

A random variable may be *qualitative* or *quantitative* in nature. The qualitative random variables yield categorical responses so that the responses fit into one category or another. For example, a response to a question such as '*Are you currently unemployed?*' would fit in the category of either '*yes*' or '*no*'. On the other hand, quantitative random variables yield numerical responses. For example, responses to questions such as, '*How many rooms are there in your house?*' or '*How many children are there in the family?*' would be in numerical

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values. Also, these values being whole numbers are considered discrete values. These are the values of *discrete quantitative random variables*. On the other hand, responses to questions like, 'How tall are you?' or 'How much do you weigh?' would be values of *continuous quantitative random variables*, since these values are measured and not counted. Some of the examples of these variables are:

### (i) Qualitative Random Variables

- Sex of students in the class.
- Political affiliation of a faculty member in the college.
- Opinions of economists regarding the economic conditions in the country.

### (ii) Quantitative Random Variables

#### (a) Discrete quantitative random variables

- Number of people attending a conference.
- Number of eggs in the refrigerator.
- Number of children at a summer camp.

#### (b) Continuous quantitative random variables

- Heights of models in a beauty contest.
- Weights of people joining a diet programme.
- Lengths of steel bars produced in a given production run.

### Check Your Progress

1. What are programming techniques? Give examples.
2. What are interpolation and extrapolation?
3. What is a model?
4. What are discrete variables?

## 1.6 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Programming techniques (or what is generally described as Operations Research or simply OR) are the model building techniques used by decision makers in modern times. They include wide variety of techniques such as linear programming, theory of games, simulation, network analysis, queuing theory and many other similar techniques.



2. Interpolation is the statistical technique of estimating, under certain assumptions, the figures missing amongst the given values of the variable itself whereas extrapolation provides figures outside the given data.
3. A model is simply a representation of a system or an object or a realistic situation in some form, formulated to explain the behaviour of the entity itself. It simplifies the real life situation which it reflects.
4. A discrete variable assumes values only within a countable or discrete range.

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### 1.7 SUMMARY

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- Quantitative methods refer to the statistical and operations research techniques that assist in the everyday decision-making process of a business/firm.
- A model is a representation of a system or an object or a realistic situation in some form, formulated to explain the behaviour of the entity itself.
- There are various types of models such as physical models, analog models and mathematical models.
- The various steps in building a model include problem formulation, construction of the model, solution of the model, testing and finally implementation.
- A symbol that retains the same value throughout a particular problem is called a constant.

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### 1.8 KEY WORDS

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- **Process control:** Application of statistical tools to industry to maintain quality of products
- **Analog model:** The analog model is a graphical or physical representation of a real object or a situation, but without the same physical appearance as the object or situation itself. The model does not look like the real system but behaves like it.
- **Linear models:** Models characterized by the premise that all components of the model have linear behaviour and relationship
- **Trade-off:** A trade-off (or tradeoff) is a situation that involves losing one quality or aspect of something in return for gaining another quality or aspect. It implies a decision to be made with full comprehension of both the upside and downside of a particular choice

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## 1.9 SELF ASSESSMENT QUESTIONS AND EXERCISES

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#### Short Answer Questions

1. Write a short note on problem formulation.
2. List some reasons for using models.
3. List some disadvantages of quantitative techniques.
4. What are quantitative random variables?

#### Long Answer Questions

1. Analyse the importance of quantitative analysis in management practices.
2. Explain the reasons for using models and the methods of developing them.

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### 1.10 FURTHER READINGS

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# UNIT 2 BASIC CONCEPT OF DIFFERENTIATION AND INTEGRATION

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*Basic Concept of  
Differentiation  
and Integration*

## NOTES

### Structure

- 2.0 Introduction
- 2.1 Objectives
- 2.2 Differentiation
- 2.3 Integration
- 2.4 Use of Differentiation for Optimisation of Business Problem
- 2.5 Statistics in Business Decision Making and Research
  - 2.5.1 Collection, Tabulation and Presentation of Data
  - 2.5.2 Measures of Central Tendency
  - 2.5.3 Graphic Representation
  - 2.5.4 Measures of Dispersion
- 2.6 Answers to Check Your Progress Questions
- 2.7 Summary
- 2.8 Key Words
- 2.9 Self Assessment Questions and Exercises
- 2.10 Further Readings

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## 2.0 INTRODUCTION

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Integration and differentiation are two fundamental concepts in calculus, which studies the change. Calculus has a wide variety of applications in many fields such as science, economy or finance, engineering, etc.

Differentiation is the algebraic procedure of calculating the derivatives. Derivative of a function is the slope or the gradient of the curve (graph) at any given point. Integration is the process of calculating either definite integral or indefinite integral.

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## 2.1 OBJECTIVES

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After going through this unit, you will be able to:

- Differentiate between differentiation and integration
- Analyse the use of differentiation for optimisation of business problems
- Understand the statistics used in business decision making and research

## 2.2 DIFFERENTIATION

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Let  $y$  be a function of  $x$ . We call  $x$  an *independent variable* and  $y$  a *dependent variable*.

**Note:** There is no sanctity about  $x$  being independent and  $y$  being dependent. This depends upon which variable we allow to take any value, and then corresponding to that value, determine the value of the other variable. Thus in  $y = x^2$ ,  $x$  is an independent variable and  $y$  a dependent, whereas the same function can be rewritten as  $x = \sqrt{y}$ . Now  $y$  is an independent variable, and  $x$  is a dependent variable. Such an 'inversion' is not always possible. For example, in  $y = \sin x + x^3 + \log x + x^{1/2}$ , it is rather impossible to find  $x$  in terms of  $y$ .

### Differential coefficient of $f(x)$ with respect to $x$

$$\text{Let } y = f(x) \quad (2.1)$$

and let  $x$  be changed to  $x + \delta x$ . If the corresponding change in  $y$  is  $\delta y$ , then

$$y + \delta y = f(x + \delta x) \quad (2.2)$$

Equations (2.1) and (2.2) imply that,

$$\delta y = f(x + \delta x) - f(x)$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

$\text{Lim}_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$ , if it exists, is called the *differential coefficient* of  $y$

with respect to  $x$  and is written as  $\frac{dy}{dx}$ .

$$\text{Thus, } \frac{dy}{dx} = \text{Lim}_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \text{Lim}_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Let  $f(x)$  be defined at  $x = a$ . The derivative of  $f(x)$  at  $x = a$  is defined as

$\text{Lim}_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$ , provided the limit exists, and then it is written as  $f'(a)$ , or

$$\left(\frac{dy}{dx}\right)_{x=a} \quad \text{We sometimes write the definition in the form } f'(a) = \text{Lim}_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

**Note:**  $f'(a)$  can also be evaluated by first finding out  $\frac{dy}{dx}$  and then putting in it,  $x = a$ .

**Notation:**  $\frac{dy}{dx}$  is also denoted by  $y'$  or  $y_1$  or  $dy$  or  $f'(x)$  in case  $y = f(x)$

**Example 2.1:** Find  $\frac{dy}{dx}$  and  $\left(\frac{dy}{dx}\right)_{x=3}$  for  $y = x^3$

**Solution:** We have  $y = x^3$ .

Let  $\delta x$  be the change in  $x$  and let the corresponding change in  $y$  be  $\delta y$ .

$$\begin{aligned} \text{Then} \quad y + \delta y &= (x + \delta x)^3 \\ \Rightarrow \quad \delta y &= (x + \delta x)^3 - y = (x + \delta x)^3 - x^3 \\ &= 3x^2 (\delta x) + 3x (\delta x)^2 + (\delta x)^3 \end{aligned}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = 3x^2 + 3x (\delta x) + (\delta x)^2$$

$$\text{Consequently,} \quad \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = 3x^2$$

$$\text{Also,} \quad \left( \frac{dy}{dx} \right)_{x=3} = 3 \cdot 3^2 = 27$$

**Example 2.2:** Show that for  $y = |x|$ ,  $\frac{dy}{dx}$  does not exist at  $x = 0$ .

**Solution:** If  $\frac{dy}{dx}$  exists at  $x = 0$ , then

$$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \text{ exists.}$$

$$\text{So,} \quad \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{f(0-h) - f(0)}{-h}$$

$$\text{Now,} \quad f(0+h) = |h|,$$

$$\text{So,} \quad \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{|h| - 0}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = 1$$

$$\begin{aligned} \text{Also,} \quad \lim_{h \rightarrow 0^+} \frac{f(0-h) - f(0)}{-h} &= \lim_{h \rightarrow 0^+} \frac{|-h| - 0}{-h} \\ &= \lim_{h \rightarrow 0^+} \frac{h}{-h} = -1 \end{aligned}$$

$$\text{Hence,} \quad \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} \neq \lim_{h \rightarrow 0^+} \frac{f(0-h) - f(0)}{-h}$$

Consequently,  $\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$  does not exist.

**Notes:** 1. A function  $f(x)$  is said to be *derivable* or *differentiable* at  $x = a$  if its derivative exists at  $x = a$ .

2. A differentiable function is necessarily continuous.

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**Proof:** Let  $f(x)$  be differentiable at  $x = a$ .

Then,  $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$  exists, say, equal to  $l$ .

$$\lim_{h \rightarrow 0} h \frac{f(a+h) - f(a)}{h} = (\lim_{h \rightarrow 0} h)l = 0$$

$$\Rightarrow \lim_{h \rightarrow 0} [f(a+h) - f(a)] = 0 \quad \Rightarrow \lim_{h \rightarrow 0} f(a+h) = f(a)$$

$$\Rightarrow \lim_{h \rightarrow 0} f(x) = f(a)$$

$h$  can be positive or negative.

In other words  $f(x)$  is continuous at  $x = a$ .

3. Converse of the statement in Note 2 is *not true* in general.

### 2.3 INTEGRATION

After learning differentiation, we now come to the 'reverse' process of it, namely integration. To give a precise shape to the definition of integration, we observe: If  $g(x)$  is a function of  $x$  such that

$$\frac{d}{dx} g(x) = f(x)$$

then we define integral of  $f(x)$  with respect to  $x$ , to be the function  $g(x)$ . This is put in the notational form as

$$\int f(x) dx = g(x)$$

The function  $f(x)$  is called the Integrand. Presence of  $dx$  is there just to remind us that integration is being done with respect to  $x$ .

For example, since  $\frac{d}{dx} \sin x = \cos x$

$$\int \cos x dx = \sin x$$

We get many such results as a direct consequence of the definition of integration, and can treat them as 'formulas'. A list of such standard results are given:

- |   |  |
|---|--|
| (1) $\int 1 dx = x$                                 | because $\frac{d}{dx} (x) = 1$   |
| (2) $\int x^n dx = \frac{x^{n+1}}{n+1} (n \neq -1)$ | because $\frac{d}{dx} \left( \frac{x^{n+1}}{n+1} \right) = x^n, n \neq -1$ |
| (3) $\int \frac{1}{x} dx = \log x$                  | because $\frac{d}{dx} (\log x) = \frac{1}{x}$                              |
| (4) $\int e^x dx = e^x$                             | because $\frac{d}{dx} (e^x) = e^x$   |
| (5) $\int \sin x dx = -\cos x$                      | because $\frac{d}{dx} (-\cos x) = \sin x$                                  |

- (6)  $\int \cos x \, dx = \sin x$  because  $\frac{d}{dx} (\sin x) = \cos x$
- (7)  $\int \sec^2 x \, dx = \tan x$  because  $\frac{d}{dx} (\tan x) = \sec^2 x$
- (8)  $\int \operatorname{cosec}^2 x \, dx = -\cot x$  because  $\frac{d}{dx} (-\cot x) = \operatorname{cosec}^2 x$
- (9)  $\int \sec x \tan x \, dx = \sec x$  because  $\frac{d}{dx} (\sec x) = \sec x \tan x$
- (10)  $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x$  because  $\frac{d}{dx} (-\operatorname{cosec} x) = \operatorname{cosec} x \cot x$
- (11)  $\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x$  because  $\frac{d}{dx} (\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
- (12)  $\int \frac{1}{1+x^2} \, dx = \tan^{-1} x$  because  $\frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2}$
- (13)  $\int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} x$  because  $\frac{d}{dx} (\sec^{-1} x) = \frac{1}{x\sqrt{1+x^2}}$
- (14)  $\frac{1}{ax+b} \, dx = \frac{\log(ax+b)}{a}$  because  $\frac{d}{dx} \left[ \frac{\log(ax+b)}{a} \right] = \frac{1}{ax+b}$
- (15)  $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{n+1} \cdot \frac{1}{a} \quad (n \neq -1)$   
because  $\frac{d}{dx} \frac{(ax+b)^{n+1}}{a(n+1)} = (ax+b)^n, n \neq -1$
- (16)  $\int a^x \, dx = \frac{a^x}{\log a}$  because  $\frac{d}{dx} a^x = a^x \log a$

One might wonder at this stage that since

$$\frac{d}{dx} (\sin x + 4) = \cos x$$

then by definition, why  $\int \cos x \, dx$  is not  $(\sin x + 4)$ ? In fact, there is nothing very sacred about number 4 and it could very well have been any constant. This suggests perhaps a small alteration in the definition.

We now define integration as:

$$\text{If } \frac{d}{dx} g(x) = f(x)$$

$$\text{then } \int f(x) \, dx = g(x) + c$$

where  $c$  is **some** constant, called the *constant of integration*. Obviously,  $c$  could have *any* value and thus integral of a function is not unique! But we could say one thing here, that *any two integrals of the same function differ by a constant*.

Since  $c$  could also have the value zero,  $g(x)$  is one of the values of  $\int f(x) \, dx$ . By convention, we will not write the constant of integration (although it is there), and thus  $\int f(x) \, dx = g(x)$ , and our definition stands.

## NOTES

The above is also referred to as **Indefinite Integral** (indefinite, because we are not really giving a *definite* value to the integral by not writing the constant of integration). We will give the definition of a definite integral also.

## NOTES

### 2.4 USE OF DIFFERENTIATION FOR OPTIMISATION OF BUSINESS PROBLEM

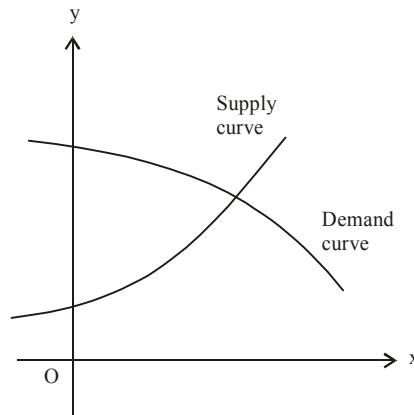
Let  $p$  be the price and  $x$  be the quantity demanded. The curve  $x = f(p)$  is called a *demand curve*. It usually slopes downwards as demand decreases when prices are increased.

Once again, let  $p$  be the price and  $x$  be the quantity supplied. The curve  $x = g(p)$  is called a *supply curve*. It is often noted that when supply is increased, the profiteers increase prices, so a supply curve frequently slopes upwards.

Let us plot the two curves on a graph paper.

If the two curves intersect, we say that an *economic equilibrium* is attained (at the point of intersection).

It is also possible that the two curves may not intersect, i.e., economic equilibrium need not always be obtained.



**Fig. 2.1** Supply and Demand Curves

#### Revenue Curves

If  $x = f(p)$  is the demand curve, and if it is possible to express  $p$  as a function of  $x$ , say  $p = g(x)$ , then function  $g$  is called inverse of  $f$ . In the problems we deal in this unit, each demand function  $f$  always possesses an inverse. So, we can write

$$x = f(p) \text{ as well as } p = g(x).$$

The product  $R$  of  $x$  and  $p$  is called the *total revenue*.

Thus,  $R = xp$  where  $x$  is the demand and  $p$  is the price. We can write  $R = p f(p) = x g(x)$ .



The *total revenue* function is defined as  $R = x g(x)$ , and if we measure  $x$  along the  $x$ -axis and  $R$  along the  $y$ -axis, we can plot the curve  $y = x g(x)$ .

This curve is called the *total revenue curve*.

### Cost Function

The cost  $c$  is composed of two parts, namely the fixed cost and the variable cost. Fixed costs are those which are not affected by the change in the amount of production. Suppose there is a publishing firm, the rent of the building in which the firm is situated is a fixed cost (whether the number of books published increases or decreases). Similarly, the salaries of the people employed is also a fixed cost (even when the production is zero).

On the other hand, the cost of printing paper is a variable cost, as, the more books they publish the more paper is required, etc.

Thus, 
$$C = VC + FC$$

where  $C$  is the total cost,  $VC$  is the variable cost and  $FC$  is the fixed cost.

### NOTES

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## 2.5 STATISTICS IN BUSINESS DECISION MAKING AND RESEARCH

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Let us analyse the statistics used in business decision making and research.

### 2.5.1 Collection, Tabulation and Presentation of Data

The statistical data may be classified under two categories depending upon the sources utilized. These categories are:

- (a) **Primary data.** Data which is collected by the investigator himself for the purpose of a specific inquiry or study is called primary data. Such data is original in character and is generated by surveys conducted by individuals or research institutions. If, for example, a researcher is interested to know what the women think about the issue of abortion, he/she must undertake a survey and collect data on the opinions of women by asking relevant questions. Such data collected would be considered as primary data.
- (b) **Secondary data.** When an investigator uses the data which has already been collected by others, such data is called *secondary data*. This data is primary data for the agency that collected it and becomes secondary data for someone else who uses this data for his own purposes. The secondary data can be obtained from journals, reports, government publications, publications of professional and research organizations, and so on. If, for example, a researcher desires to analyse the weather conditions of different regions, he can get the required information or data from the records of the meteorology department. Even though secondary data is less expensive to collect in terms of money and time, the quality of this data may even be

better under certain situations because it may have been collected by persons who were specifically trained for that purpose. It is necessary to critically investigate the validity of the secondary data as well as the credibility of the primary data collection agency.

## NOTES

### 2.5.2 Measures of Central Tendency

The data we collect can be more easily understood if it is presented graphically or pictorially. Diagrams and graphs give visual indications of magnitudes, groupings, trends and patterns in the data. These important features are more simply presented in the form of graphs. Also, diagrams facilitate comparisons between two or more sets of data.

The diagrams should be clear and easy to read and understand. Too much information should not be shown in the same diagram; otherwise, it may become cumbersome and confusing. Each diagram should include a brief and a self-explanatory title dealing with the subject matter. The scale of the presentation should be chosen in such a way that the resulting diagram is of appropriate size. The intervals on the vertical as well as the horizontal axis should be of equal size; otherwise, distortions would occur.

Diagrams are more suitable to illustrate the data which is discrete, while continuous data is better represented by graphs. The following are the diagrammatic and graphic representation methods that are commonly used.

#### 1. Diagrammatic Representation

- (a) Bar Diagram                      (b) Pie Chart                      (c) Pictogram

#### 2. Graphic Representation

- (a) Histogram                      (b) Frequency Polygon  
(c) Cumulative Frequency Curve (Ogive)

Each of these is briefly explained and illustrated below.

#### Diagrammatic Representation

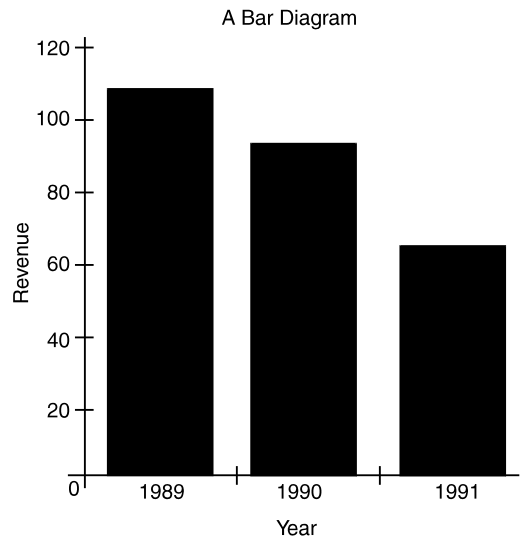
**(a) Bar Diagram.** Bars are simple vertical lines where the lengths of the bars are proportional to their corresponding numerical values. The width of the bar is unimportant, but all bars should have the same width so as not to confuse the reader. Additionally, the bars should be equally spaced.

**Example 2.3:** Suppose that the following were the gross revenue (in \$100,000.00) of a company XYZ for the years 1989, 1990 and 1991.

Year	1989	1990	1991
Revenue	110	95	65

Construct a bar diagram for this data.

**Solution:** The bar diagram for this data can be constructed as follows with the revenues represented on the vertical axis ( $Y$ -axis) and the years represented on the horizontal axis ( $X$ -axis).



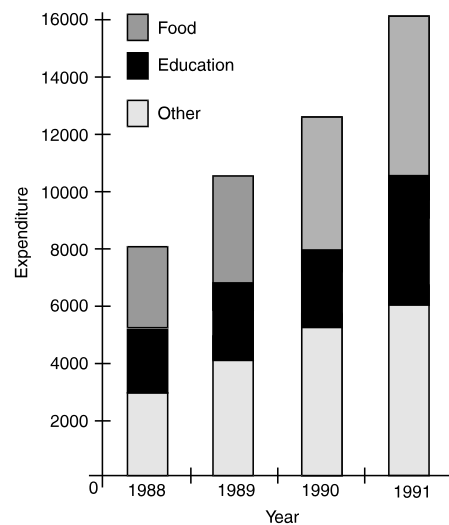
## NOTES

The bars drawn can be further subdivided into components depending upon the type of information to be shown in the diagram. This will be clear by the following example in which we are presenting three different components in a bar.

**Example 2.4:** Construct a subdivided bar chart for the three types of expenditures in dollars for a family of four for the years 1988, 1989, 1990 and 1991 as given below:

Years	Food	Education	Other	Total
1988	3000	2000	3000	8000
1989	3500	3000	4000	10500
1990	4000	3500	5000	12500
1991	5000	5000	6000	16000

**Solution:** The subdivided bar chart would be as follows:



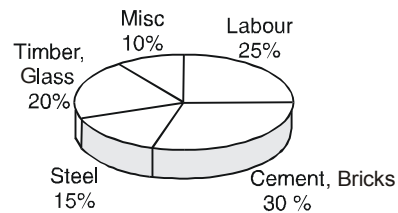
## NOTES

**(b) Pie Chart.** This type of diagram enables us to show the division of a total into its component parts. The diagram is in the form of a circle and is also called a *pie* because the entire diagram looks like a pie and the components resemble slices cut from it. The size of the slice represents the proportion of the component out of the whole.

**Example 2.5:** The following figures relate to the cost of the construction of a house. The various components of cost that go into it are represented as percentages of the total cost.

Items	Labour	Cement, Bricks	Steel	Timber, Glass	Miscellaneous
% Expenditure	25	30	15	20	10

**Solution:** The pie chart for this data is presented as follows:



Pie charts are very useful for comparison purposes, especially when there are only a few components. If there are too many components, it may become confusing to differentiate the relative values in the pie.

**(c) Pictogram:** Pictogram means presentation of data in the form of pictures. It is quite a popular method used by governments and other organizations for informational exhibition. Its main advantage is its attractive value. Pictograms stimulate interest in the information being presented.

News magazines use pictogram very often for representing data. For comparing the strength of the armed forces of the USA and Russia, for example, they will simply show sketches of soldiers where each sketch may represent 100,000 soldiers. Similar comparison for missiles and tanks is also done.

### 2.5.3 Graphic Representation

**(a) Histogram.** A histogram is the graphical description of data and is constructed from a frequency table. It displays the distribution method of a data set and is used for statistical as well as mathematical calculations.

The word histogram is derived from the Greek word *histos* which means 'anything set upright' and *gramma* which means 'drawing, record, writing'. It is considered as the most important basic tool of statistical quality control process.

In this type of representation the given data are plotted in the form of a series of rectangles. Class intervals are marked along the  $X$ -axis and the

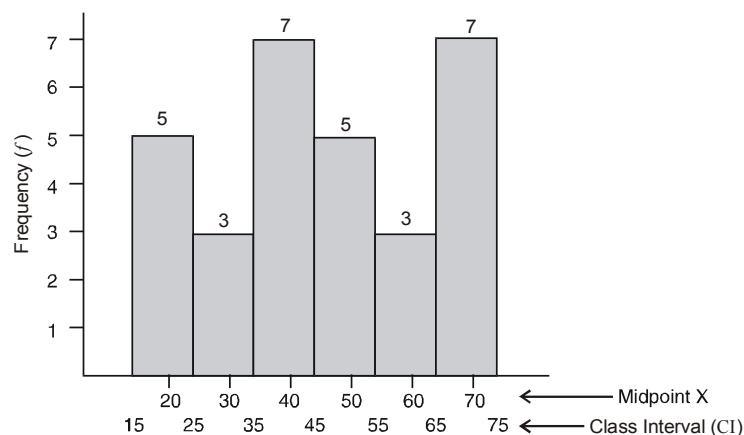
## NOTES

frequencies along the  $Y$ -axis according to a suitable scale. Unlike the bar chart, which is one-dimensional, meaning that only the length of the bar is important and not the width, a histogram is two-dimensional in which both the length and the width are important. A histogram is constructed from a frequency distribution of a grouped data where the height of the rectangle is proportional to the respective frequency and the width represents the class interval. Each rectangle is joined with the other and any blank spaces between the rectangles would mean that the category is empty and there are no values in that class interval.

As an example, let us construct a histogram for the previous example of ages of 30 workers. For convenience sake, we will present the frequency distribution along with the midpoint of each interval, where the midpoint is simply the average of the values of the lower and upper boundary of each class interval. The frequency distribution table is shown as follows:

Class Interval (CI) (years)	Midpoint ( $X$ )	Frequency ( $f$ )
15 and upto 25	20	5
25 and upto 35	30	3
35 and upto 45	40	7
45 and upto 55	50	5
55 and upto 65	60	3
65 and upto 75	70	7

The histogram of this data is shown as follows:

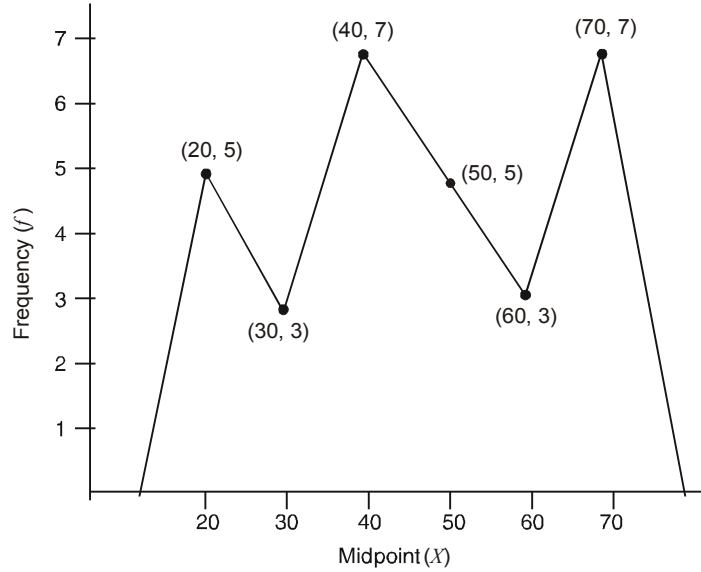


**(b) Frequency Polygon.** A frequency polygon is a line chart of frequency distribution in which either the values of discrete variables or midpoints of class intervals are plotted against the frequencies and these plotted points are joined together by straight lines. Since the frequencies generally do not start at zero or end at zero, this diagram as such would not touch the horizontal axis. However, since the area under the entire curve is the same as that of a histogram which is 100 per cent of the data presented, the curve can be enclosed such that the starting point is joined with a fictitious preceding point whose value is zero, so

**NOTES**

that the start of the curve is at horizontal axis and the last point is joined with a fictitious succeeding point whose value is also zero, such that the curve ends at the horizontal axis. This enclosed diagram is known as the frequency polygon.

We can construct the frequency polygon based on the table mentioned above as follows:

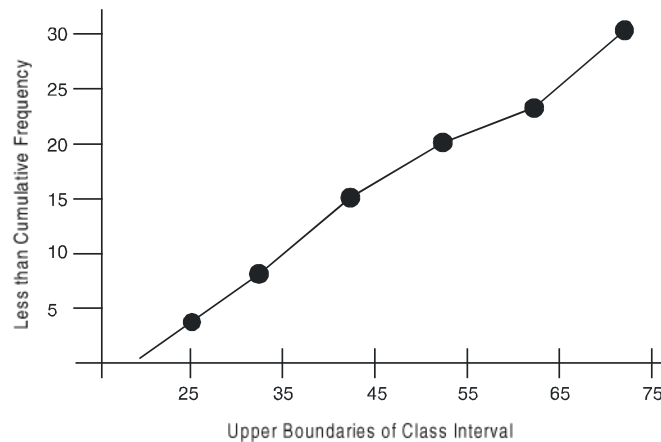


**(c) Cumulative Frequency Curve (Ogives).** The cumulative frequency curve or ogive is the graphic representation of a cumulative frequency distribution. Ogives are of two types. One of these is *less than* and the other one is *greater than* ogive. Both these ogives are constructed based upon the following table of our example of 30 workers.

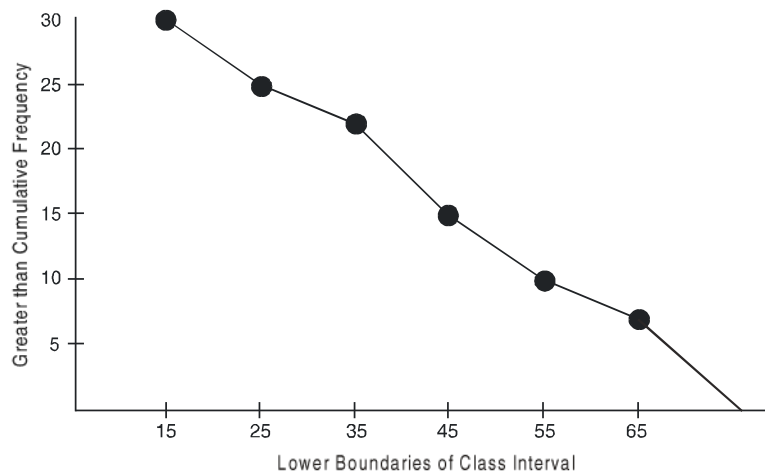
Class Interval (CI) (years)	Midpoint (X)	Frequency (f)	Cumulative Frequency (less than)	Cumulative Frequency (greater than)
15 and upto 25	20	5	5 (less than 25)	30 (more than 15)
25 and upto 35	30	3	8 (less than 35)	25 (more than 25)
35 and upto 45	40	7	15 (less than 45)	22 (more than 35)
45 and upto 55	50	5	20 (less than 55)	15 (more than 45)
55 and upto 65	60	3	23 (less than 65)	10 (more than 55)
65 and upto 75	70	7	30 (less than 75)	7 (more than 65)

(i) *Less than ogive.* In this case *less than* cumulative frequencies are plotted against upper boundaries of their respective class intervals.

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(ii) *Greater than ogive*. In this case *greater than* cumulative frequencies are plotted against the lower boundaries of their respective class intervals.



These ogives can be used for comparison purposes. Several ogives can be drawn on the same grid, preferably with different colours for easier visualization and differentiation.

Although, diagrams and graphs are a powerful and effective media for presenting statistical data, they can only represent a limited amount of information and they are not of much help when intensive analysis of data is required.

### 2.5.4 Measures of Dispersion

A measure of dispersion, or simply *dispersion* may be defined as statistics signifying the extent of the scatteredness of items around a measure of central tendency.

A measure of dispersion may be expressed in an 'absolute form', or in a 'relative form'. It is said to be in an absolute form when it states the actual amount by which the value of an item on an average deviates from a measure of central tendency. Absolute measures are expressed in concrete units, i.e., units in terms of which the data have been expressed, e.g., rupees, centimetres, kilograms, etc. and are used to describe frequency distribution.

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A relative measure of dispersion is a quotient obtained by dividing the absolute measures by a quantity in respect to which absolute deviation has been computed. It is as such a pure number and is usually expressed in a percentage form. Relative measures are used for making comparisons between two or more distributions.

A measure of dispersion should possess the following characteristics which are considered essential for a measure of central tendency.

- (a) It should be based on all observations.
- (b) It should be readily comprehensible.
- (c) It should be fairly and easily calculated.
- (d) It should be affected as little as possible by fluctuations of sampling.
- (e) It should be amenable to algebraic treatment.

The following are the common measures of dispersion:

(i) The range, (ii) The semi-interquartile range or the quartile deviation, (iii) The mean deviation, and (iv) The standard deviation. Of these, the standard deviation is the best measure.

### Check Your Progress

1. What is a primary data?
2. What is a pictogram?
3. Define a frequency polygon.

## 2.6 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Data which is collected by the investigator himself for the purpose of a specific inquiry or study is called primary data.
2. Pictogram means presentation of data in the form of pictures. It is quite a popular method used by governments and other organizations for informational exhibition.
3. A frequency polygon is a line chart of frequency distribution in which either the values of discrete variables or midpoints of class intervals are plotted against the frequencies and these plotted points are joined together by straight lines.

## 2.7 SUMMARY

- Integration and differentiation are two fundamental concepts in calculus, which studies the change. Calculus has a wide variety of applications in many fields such as science, economy or finance, engineering, etc.



- Differentiation is the algebraic procedure of calculating the derivatives. Derivative of a function is the slope or the gradient of the curve (graph) at any given point. Integration is the process of calculating either definite integral or indefinite integral.
- A measure of dispersion, or simply dispersion may be defined as statistics signifying the extent of the scatteredness of items around a measure of central tendency.
- The measures of dispersion are useful in determining how representative the average is as a description of the data, in comparing two or more series with regard to their scatter.
- Besides collecting the appropriate data, a significant emphasis is also laid on the suitable representation of that data. This calls for using several data representation techniques, which depend on the nature and type of data collected.
- Data can be represented through frequency distributions, bar diagrams, pictograms, histograms, frequency polygons, ogives, etc.

## NOTES

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### 2.8 KEY WORDS

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- **Quartile deviation:** A type of range based on the quartiles.
- **Mean deviation:** The arithmetic mean of the absolute deviations of a series of values.
- **Standard deviation:** The measure of the dispersion of a set of values and is calculated from the mean of squared deviations.

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### 2.9 SELF ASSESSMENT QUESTIONS AND EXERCISES

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#### Short Answer Questions

1. Differentiate between an independent and dependent variable.
2. State the functions of integration and differentiation.
3. State in brief the two methods of data collection.

#### Long Answer Questions

1. Explain the use of differentiation for optimization of business problem.
2. Find the mean deviation about the mean of the following data of ages of married men in a certain town.

Ages	15-24	25-34	35-44	45-54	55-64	65-74
No. of Men	33	264	303	214	128	58

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3. The following figures give the income of 10 persons in rupees. Find the standard deviation.

114, 115, 123, 120, 110, 130, 119, 118, 116, 115

4. Calculate the mean and standard deviation of the following values of the world's annual gold output millions of pound (in for 20 different years)

94, 95, 96, 93, 87, 79, 73, 69, 68, 67, 78, 82, 83, 89, 95, 103, 108, 117, 130, 97.

Also calculate the percentage of cases lying outside the mean at distances  $\pm s$ ,  $\pm 2s$ ,  $\pm 3s$  where  $s$  denotes standard deviation.

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## 2.10 FURTHER READINGS

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Taha, H.A. 2006. *Operations Research: An Introduction*, 8th edition. New Delhi: Prentice-Hall of India.

Jensen, Paul A., and Jonathan F. Bard. 2003. *Operations Research Models and Methods*. New York: John Wiley & Sons.

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## UNIT 3 VARIABLES AND FUNCTION

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### Structure

- 3.0 Introduction
- 3.1 Objectives
- 3.2 Graphical Representation of Functions and their Applications in Cost and Revenue Behaviour
  - 3.2.1 Inclination and Slope
- 3.3 Marginal Costing and Elasticity of Demands
  - 3.3.1 Maxima and Minima of Functions
- 3.4 Answers to Check Your Progress Questions
- 3.5 Summary
- 3.6 Key Words
- 3.7 Self Assessment Questions and Exercises
- 3.8 Further Readings

### NOTES

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### 3.0 INTRODUCTION

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A real-valued function of a real variable is a function that takes as input of a real number, commonly represented by the variable  $x$ , for producing another real number, the value of the function, commonly denoted  $f(x)$ .

In this unit, you will be able to discuss the linear and non-linear variables, graphical representation of functions and their application. You will also learn about the elasticity of demand and the decisions on minimizing and maximizing costs and outputs.

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### 3.1 OBJECTIVES

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After going through this unit, you will be able to:

- Describe functions and variables
- Differentiate between linear and non-linear functions
- Discuss slope and its relevance
- Describe the use of functional relationships and analyse elasticity of demand

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### 3.2 GRAPHICAL REPRESENTATION OF FUNCTIONS AND THEIR APPLICATIONS IN COST AND REVENUE BEHAVIOUR

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In economics and other sciences we have to deal with more than one variable. The changes in any variable are not of great importance unless they are associated

## NOTES

with changes in related variables. The way in which one variable depends on other variables is described by means of *functions* which pervade all mathematics and its applications. A function is a relation which associates any given number with another number. Functions can be defined in several ways.

The expression  $y = f(x)$  is a general statement of a function. The actual mapping is not explicit here.

Constant functions, polynomial functions and relation functions are one class of functions.

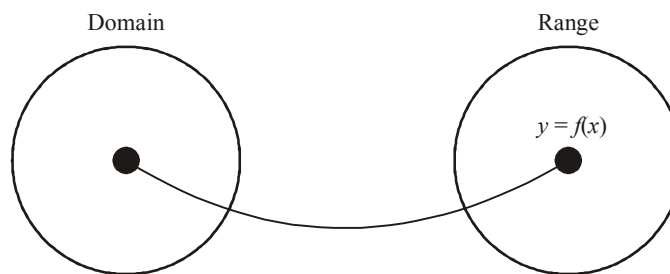
We define a function from the set  $X$  into the set  $Y$  as a set of ordered pairs  $(x, y)$  where  $x$  is an element of  $X$  and  $y$  is an element of  $Y$  such that for each  $x$  in  $X$  there is only one ordered pair  $(x, y)$  in the function  $f$ . The notation used is

$$f: X \rightarrow Y \text{ or } x \rightarrow f(x) \text{ or } y = f(x)$$

A function is a mapping or transformation of  $x$  into  $y$  or  $f(x)$ . The variable  $x$  represents elements of the *domain* and is called the *independent* variable. The variable  $y$  representing elements of the *range* is called the *dependent* variable. The function  $y = f(x)$  is often called a *single valued function* since there is a unique  $y$  in the range for each specified  $x$ . The converse may not necessarily be true.

$y = f(x)$  is the *image* of  $x$

A function whose domain and range are sets of real numbers is called a *real-valued* function of a real variable.



A function is a *constant* function if the range consists of a single element. It may be written  $y = k$  or  $f(x) = k$  where  $k$  is a constant.

If  $f(x) = x$  for all  $x$  it is an *identity* function.

$f(x) = |x|$  is the *absolute value* function.

Often a function depends on several independent variables. If there are  $n$  independent variables  $x_1, x_2, \dots, x_n$  and the range is the set of all possible values of  $y$  corresponding to the domain of  $(x_1, x_2, \dots, x_n)$  we say that  $y$  is a function of  $x_i$ 's:  $y = f(x_1, x_2, \dots, x_n)$ . Letters other than  $f$  may be used to represent a function.

*Equality of functions.* Two functions  $f$  and  $g$  are equal if and only if they have the same domain and/or  $f(x) = g(x)$  for all  $x$  in the domain.

**Example 3.1:** If  $p$  is the price and  $x$  the quantity of a commodity demanded by the consumers, we write the demand function:

$$x = f(p)$$

**Example 3.2:** If  $S$  stands for saving and  $Y$  for income, then

$$S = f(Y)$$

conveys that saving depends on income.

**Example 3.3:** If saving depends on income and also on rate of interest  $r$ ; then saving is a function of two variables  $r, Y$ .

$$S = f(r, Y)$$

**Example 3.4:** If  $u$  is a function of three variables  $x, y, z$ , then we write

$$u = f(x, y, z)$$

**Example 3.5:** If production ( $x$ ) is a function of land ( $D$ ), labour ( $L$ ), capital ( $K$ ), organization ( $O$ ), then we can write

$$x = f(D, L, K,$$

$O)$  as the production function.

**Example 3.6:** If utility is a function of  $n$  commodities whose quantities are represented by  $x_1, x_2, \dots, x_n$ , then the utility function can be written as

$$U = U(x_1, x_2, \dots, x_n)$$

### Explicit and Implicit Functions

If one variable is expressed directly in terms of other variables, it is a case of an explicit function.

**Example 3.7:**  $y = 3x + 2, y = Ax^2 + 9$  are explicit functions.

If the relation between the variables is given by an equation containing all the variables, we have an implicit function which may not distinguish between dependent and independent variables.

**Example 3.8:**  $x^2 + 7xy + 9y^2 + 8 = 0, y - x \log y = x^3, f(x, y) = 0$  are implicit functions.

### Graphs of Functions

The graph of a function  $f(x)$  is a set of points in the plane.

We can picture a function  $y = f(x)$  as a graph showing the relation between  $x$  and  $y$ .

Every point  $(x, y)$  which satisfies the equation  $y = f(x)$  is on the curve.

The coordinates of every point  $(x, y)$  on the curve satisfy the equation  $y = f(x)$ . Thus the graph of a function consists of all points in the form  $[x, f(x)]$  where  $x$  is a number in the range of  $f(x)$ .

## NOTES

**Even and Odd Functions**

A function  $f(x)$  is said to be an *even* function if for every  $x$  in a certain range,

$$f(-x) = f(x)$$

**NOTES**

A function  $f(x)$  is an *odd* function if for every  $x$  in a certain range,

$$f(-x) = -f(x)$$

**Example 3.9:**  $f(x) = x^2$  is an even function.

**Example 3.10:**  $f(x) = x^3$  is an odd function.

**Single and Multivalued Functions**

If to each value of  $x$  there corresponds one and only one value of  $y$ , the function is said to be single valued or uniform, e.g.,

$$y = x(x - 1)$$

If more than one value of  $y$  corresponds to one value of  $x$ , the function is said to be multivalued or multiform, e.g.,

$$y = \sqrt{x}$$

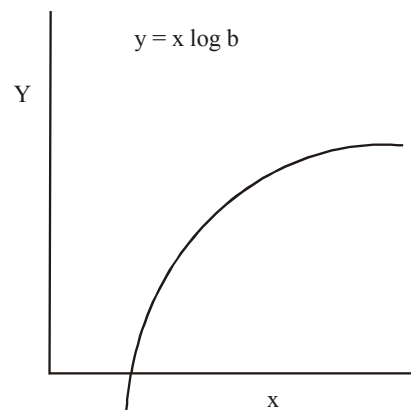
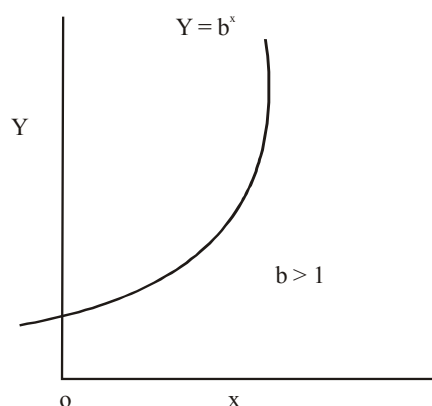
This is a two valued function because any value of  $x$ , say 4, gives two values of  $y$  i.e., + 2 and - 2.

**Algebraic and Non-Algebraic Functions**

Algebraic functions are obtained through a finite number of algebraic operation like addition, subtraction, multiplication, division and through solving a finite number of algebraic equations. Polynomials, rational and radical functions are algebraic.

Algebraic and non-algebraic functions are another classification of functions. Any function expressed in terms of polynomials and/or roots of polynomials is an algebraic function. The functions so far dealt are all algebraic.

Non-algebraic functions are exponential, logarithmic, trigonometric and so on. In an exponential function, the independent variable appears as the exponent as in the function:  $Y = b^x$ . A logarithmic function may be as:  $y = x \log b$ . Non-algebraic functions are also known as transcendental functions.



## Polynomial Function

A *polynomial* function is of the form

$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$  where  $n$  is a positive integer and  $a_0, a_1, \dots, a_n$  are real numbers.

This is a polynomial function of degree  $n$  in  $x$ .

Polynomial Function is a multi-term function. The general form of a single variable,  $x$ , polynomial function is:  $y = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \dots + a_nx^n$ . Each element contains a coefficient as well as a non-negative integer power of variables. [The first two terms can be written as  $a_0 + a_1x$ , since,  $x_0 = 1$  and  $x^1$  is commonly written as  $x$ ].

Depending on the value of the integer ' $n$ ' (which specifies the highest power of  $x$ ), several subclasses of polynomial functions emerge:

Case of  $n = 0$ :  $y = a_0$  [Constant function]

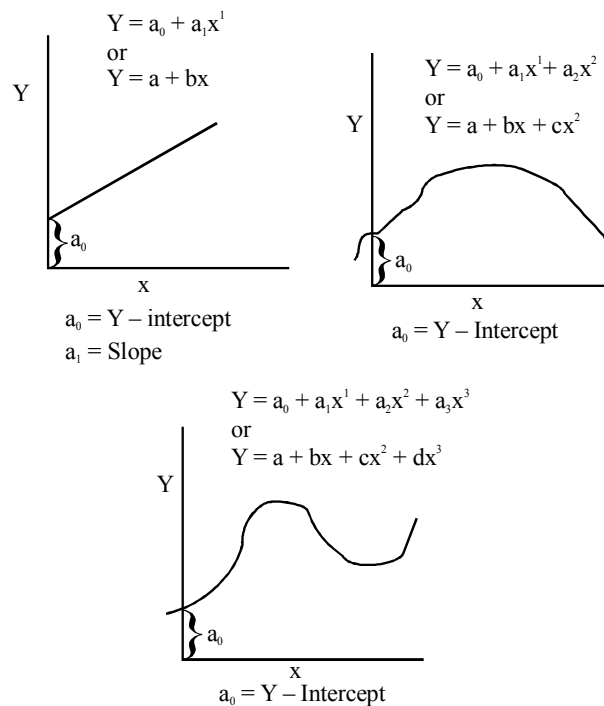
Case of  $n = 1$ :  $y = a_0 + a_1x^1$  [Linear function]

Case of  $n = 2$ :  $y = a_0 + a_1x^1 + a_2x^2$  [Quadratic function]

Case of  $n = 3$ :  $y = a_0 + a_1x^1 + a_2x^2 + a_3x^3$  [Cubic function]

We can proceed further assigning other values to ' $n$ '. The powers of  $x$  are called exponents. The highest power involved, i.e., the value of ' $n$ ', is often called the degree of the polynomial function. A cubic function, with  $n = 3$ , is a third-degree polynomial.

Graphs below give the linear, quadratic and cubic functions respectively:



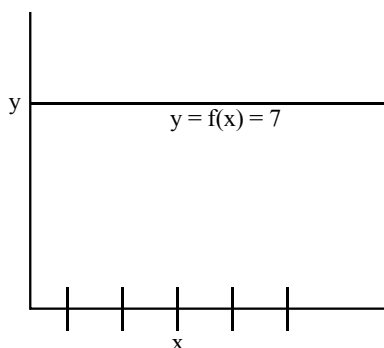
## NOTES

### Constant Functions

A zero degree polynomial function is a constant function.

#### NOTES

Constant Function takes only one value as its range.  $y = f(x) = 7$ ; or  $y = 7$ ; or  $f(x) = 7$ . Regardless of the value of  $x$ , value of  $y = 7$ . Value of  $y$  is, perhaps exogenously determined. Such a function, in the coordinate plane, will appear as a horizontal straight line. Graph below gives the constant function.



**Example 3.11:**  $f(x) = 3$ ,  $g(x) = p$ ,  $h(x) = -k$  are constant functions.

### Linear Functions

A polynomial function of degree 1 is a linear function.

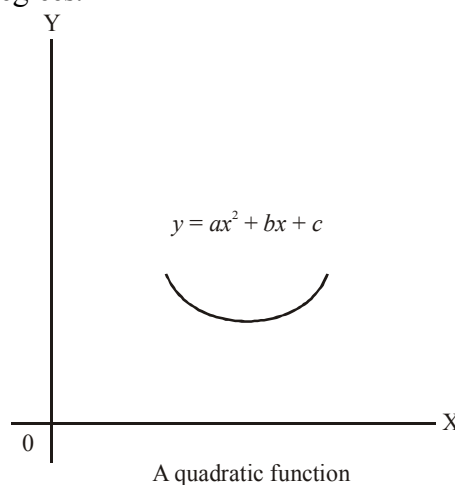
**Example 3.12:**  $f(x) = mx + c$  is a linear function.

The linear function  $f(x) = x$  is called the *identity* function.

The graph of a linear function is a straight line.

### Quadratic Functions

A polynomial function of degree 2 is a quadratic function. We can similarly have functions of higher degrees.



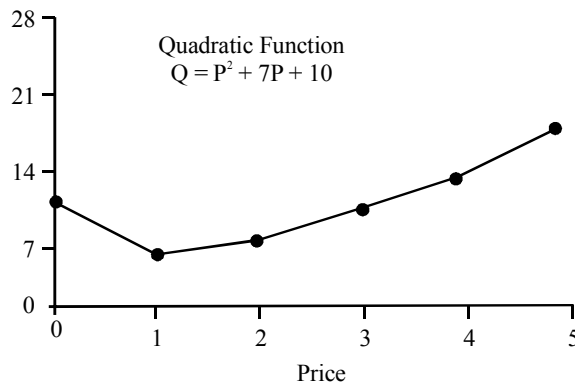
Let the demand for a product be given by the quadratic function:  $Q = P^2 - 7P + 10$ , where  $P$  is price.



We can get the graph plotted by first mapping the function in the form of a table as below. As P cannot take negative value, we take it as zero first and allow it to rise a unit at every step and relevant Q values got thus are:

P	0	1	2	3	4	5
Q	10	4	7	12	19	28

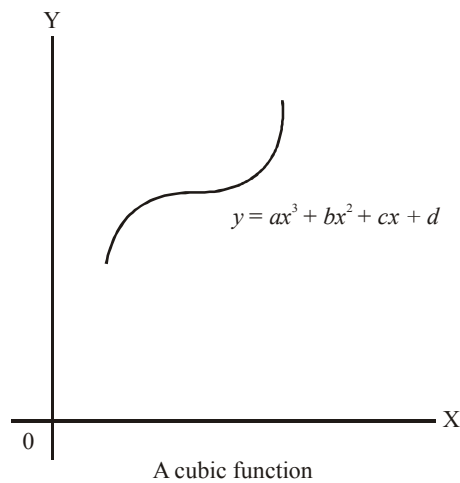
**NOTES**



Graph above depicts the function  $Q = P^2 - 7P + 10$ . The curve does not conform to normal demand curve, generally speaking. Perhaps this is vanity demand curve. A quadratic function is different from a quadratic equation. A quadratic equation results when a quadratic function is set to zero. That is, the quadratic function,  $P^2 - 7P + 10$  if made equal to zero, i.e.,  $P^2 - 7P + 10 = 0$ , results in a quadratic equation. For the quadratic equation the roots can be worked out.

**Example 3.13:**  $f(x) = ax^2 + bx + c$  is a quadratic function. Its graph is a parabola.

**Example 3.14**  $f(x) = 3x^3 - lx^2 + 2x - 1$  is a cubic function.



**Rational Functions**

A rational function is of the form

**NOTES**

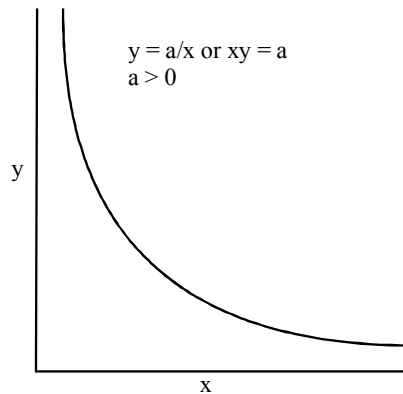
$$y = \frac{f(x)}{g(x)} \text{ where } f(x), g(x) \text{ are polynomials and } g(x) \neq 0$$

A rational function is one which is expressed as a ratio of two polynomials in the variable  $x$ . Look at the function below:

$$y = \frac{x - 5}{x^2 + 2x + 20}$$

It is a polynomial in which  $y$  is expressed as a ratio of  $x - 5$  to  $x^2 + 2x + 20$  (both are polynomials), is a rational function. Any polynomial function is a relational function, because it can be always expressed as a ratio to 1, which is a constant function.

A special rational function that has quite an interesting application in business is the function:  $y = a/x$  or  $xy = a$ . This function plots as a rectangular hyperbola as shown in the graph below.



Since the product of the two terms is always a given constant, this function may be used to represent average fixed cost curve, a special demand curve where total expenditure (i.e., price  $\times$  quantity) is always the same.

The rectangular hyperbola drawn for  $xy = 1$ , never meets the axes, for whatever levels of upward and horizontal extensions.

**Example 3.15:**  $y = \frac{2x - 1}{2x^2 + 2x - 5}$  is a rational function.

**Functions of more than One Independent Variable**

Functions can be classified on the basis of number of independent variables. So far only one independent variable,  $x$ , was dealt by us. Instead two, three or any number of independent variables may be involved. We know production is a function of labour (L) and capital (K). So, a production function may be presented as:  $Q =$

$f(K, L)$ . Consumer utility can be given as a function of 3 different commodities and the function is  $y = f(u, v, w)$ .

Functions of more than one variable can be constant, linear or nonlinear. Look at these forms:

$$Y = a_1 + a_2 + a_3 + a_4 \dots \dots \dots a_n \quad \text{(Constant function)}$$

$$Y = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 \dots \dots \dots a_n x_n \quad \text{(Linear function)}$$

$$Y = a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2 \quad \text{(Quadratic function)}$$

$$Y = a_1 x_1^3 + a_2 x_1^2 x_2 + a_3 x_1 x_2^2 + a_4 x_1 x_2^3 \quad \text{(Cubic function)}$$

As presented already, linear functions are 1st degree polynomial. These could have single or more independent variables. A single variable linear function runs like:

$Y = a + bX$ . A two variable linear function is:

$$Y = a + bx_1 + cx_2.$$

**NOTES**

**3.2.1 Inclination and Slope**

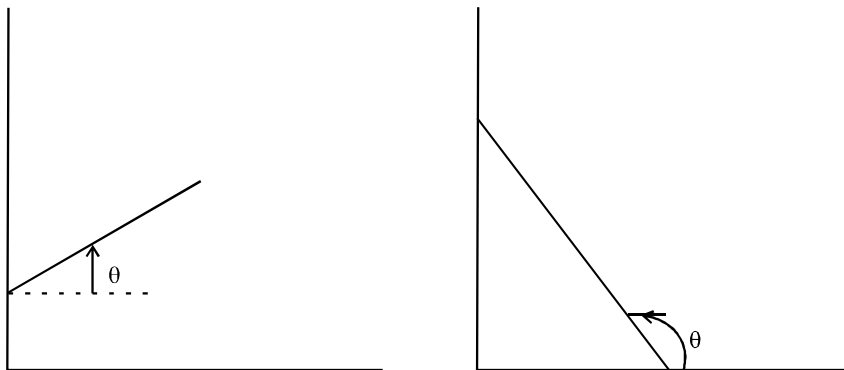
If a line  $AP$  is not horizontal, its angle of inclination is the least positive angle through which the positive  $x$ -axis must be rotated to coincide with  $P$ .  $\theta$  is the angle of inclination of  $AP$ . If  $AP$  is horizontal,  $\theta = 0$ . If  $AP$  is vertical  $\theta = 90^\circ$ .

The slope of a line  $PQ$  is defined by  $m = \tan \theta$ .

Slope refers to the inclination of a line to the  $x$ -axis. It gives a measure of the rate of change in dependent variable,  $y$ , for a unit change in independent variable,  $x$ . Both the direction and quantity of change produced are indicated by slope.

Slope of a line can be studied in five ways.

- (i) Slope is the tangent of the angle made by the line with the  $x$ -axis when we move anti-clockwise from the  $x$ -axis to the line. If  $\theta$  is the angle that the line makes with the positive direction of  $x$ -axis, then Slope of the line = Tangent = Length of Opposite side/Length of Adjacent side. See the graphs below.



- (ii) Slope of a line can be measured through ordinates and coordinates. If  $(x_1, y_1)$  and  $(x_2, y_2)$  are any two points on the line, then slope is given by:  $(y_2 - y_1) / (x_2 - x_1)$ .

## NOTES

- (iii) When an equation of a line is given as,  $y = a + bx$ , then slope is given by 'b' in that equation.
- (iv) Slope is nothing but the regression coefficient of  $y$  on  $x$  or  $b_{yx} = \frac{\sum x_i y_i}{\sum x_i^2}$  where  $x_i$  is deviation of  $X$  values from their Mean and  $y_i$  is the deviation of  $Y$  values from their Mean.
- (v) Slope is also studied by amount of change in  $y$ ,  $\Delta y$  divided by amount of change in  $x$ ,  $\Delta x$ . Slope =  $\Delta y / \Delta x$ .

**Example 3.16:**

- Suppose three lines make each an angle of (i)  $30^\circ$ , (ii)  $45^\circ$  and  $60^\circ$  with the positive direction of  $x$ -axis. Then the slope of the lines are:
  - $\tan 30^\circ = 1/\sqrt{3}$
  - $\tan 45^\circ = 1$ ;  $\tan 60^\circ = \sqrt{3}$
- Suppose two points on a line are:  $2, -4$  and  $1, 7$ . Its slope =  $(y_2 - y_1)/(x_2 - x_1) = 7 - (-4)/1 - 2 = 11/-1 = -11$ . The line has negative slope.
- A machine costs ₹ 1,00,000. 5 years after, its value falls to ₹ 60,000. If value is a linear function of time, find the depreciation function.

**Solution:** Let the value of the machine at year ' $t$ ' be:  $v = a + bt$ . Put  $v = 1,00,000$  at  $t = 0$  and  $v = 60000$  at  $t = 5$ . We get the following two equations:

$$(i) \quad 1,00,000 = a + b(0) \text{ or } a = 1,00,000 \quad \dots (1)$$

$$(ii) \quad 60000 = a + b(5) \text{ or } a + 5b = 60000 \quad \dots (2)$$

Solving equation (1) and (2) we get,  $5b = -40000$  or  $b = -8000$

The annual rate of depreciation is 8000 and the value of the machine falls by ₹ 8000 p.a.

**Linear Cost Function and Slope Thereof**

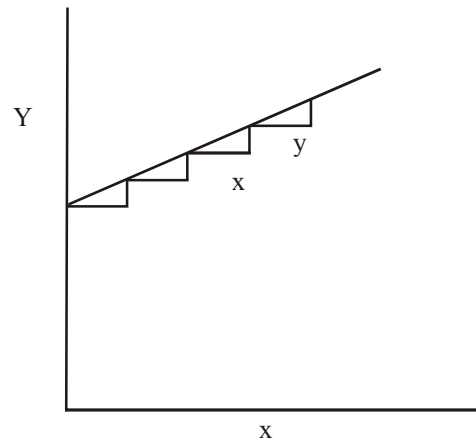
Total cost (TC) =  $1500 + 400Q$ , where  $Q$ ' is the quantity produced, and the constant term is 1500 being the fixed cost and coefficient of  $Q$  is 400, which is the slope of the curve indicating the change in total cost, if output rises or falls by one unit. In the case of a linear function, the slope is constant at all points on the curve.

Slope is a highly relevant concept in our analysis. Slope measures the rate of change in the dependent variable for a unit change in the independent variable. This is given by the tangent of the curve at a point.

$$\text{Tangent A} = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{\Delta Y}{\Delta X}$$

In the following graph, the slope is presented by slope =  $\Delta Y / \Delta X$

Cost Curve and Slope

**NOTES**

We have standardized,  $\Delta x$  as 1 unit. Then  $\Delta y$ , the vertical side of the shaded triangle is the slope. And it is constant, for all the four triangles depicted.

Actually slope is the regression coefficient in the regression line. Suppose  $x$  as the quantity and  $y$ , the total cost. Let the details be as follows.

						<b>Total</b>	<b>Mean</b>
$X$	2	3	5	8	2	20	6
$Y$	2300	2700	3500	4700	6300	17500	3500
$X - \text{Mean } X = X$	-4	-3	-1	2	6	0	-
$Y - \text{Mean } Y = Y$	-1600	-1200	-400	800	2400	0	-
$xy$	6400	3600	400	1600	14400	26400	-
$x^2$	16	9	1	4	36	66	-

The regression of  $X$  on  $Y$ :  $Y = a + bX$

The value of ' $b$ ' =  $\Sigma xy / \Sigma x^2 = 26400/66 = 400$ . This is the slope of the curve.

$$a = \text{Mean of } Y - b(\text{Mean of } X) = 3500 - 400(6) = 1500$$

So, the regression equation is:  $Y = 1500 + 400X$ . Actually this equation is the same as the  $TC = 1500 + 400Q$ , with which we started. Graph (i) gives this.

Slope of a curve can also be found by differentiation. We know,  $TC = 1500 + 400Q$ . Differentiate  $TC$  with respect to  $Q$  and we will get the slope.

$$dTC/dQ = 400.$$

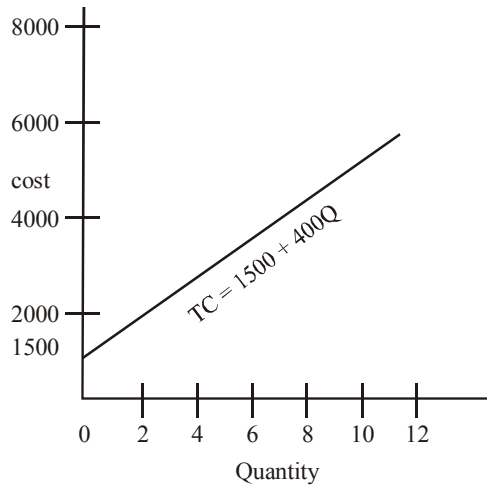
A linear function of the type  $Y = bX$  is also possible with no intercept. Suppose the following pattern of  $X$  and  $Y$ :

$X$ :	1	2	3	4	5
$Y$ :	100	200	300	400	500

This pattern also depicts a straight line, but its  $Y$  intercept is zero, in other words, the line passes through the origin. Graph (ii) gives an account of the same.

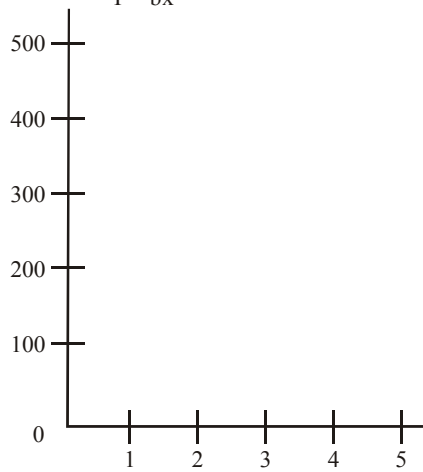
**NOTES**

Cost curve and slope  
 $Y = a + bx$



**Graph (i)**

Cost curve and slope  
 $Y = bx$



**Graph (ii)**

**Linear Demand Function and Slope Thereof**

Suppose the quantity demanded and price values are linearly related and the demand function is as follows:

Unit price	P:	4	5	6	7	8
Quantity	Q:	1400	1200	1000	800	600

We are interested in finding the rate of change in Q for a unit change in P. You know we have made this rate of change as 200. We can work it out through the regression equation, as well as the differentiation route.

**Example 3.17:**

						Total	Mean
P	4	5	6	7	8	30	6
Q	1400	1200	1000	800	600	5000	1000
P - Mean P = $p$	-2	-1	0	1	2	0	-
Q - Mean Q = $q$	400	200	0	-200	-400	0	-
$pq$	-800	-200	0	-200	-800	-2000	-
$p^2$	4	1	0	1	4	10	-

First the regression model is attempted.

The slope or regression coefficient =  $b = \frac{\sum pq}{\sum p^2} = \frac{-2000}{10} = -200$ .

$$\begin{aligned} \text{The constant} &= \text{Mean of } Q - b (\text{Mean of } P) \\ &= 1000 - (-200 \times 6) = 1000 + 1200 = 2200 \end{aligned}$$

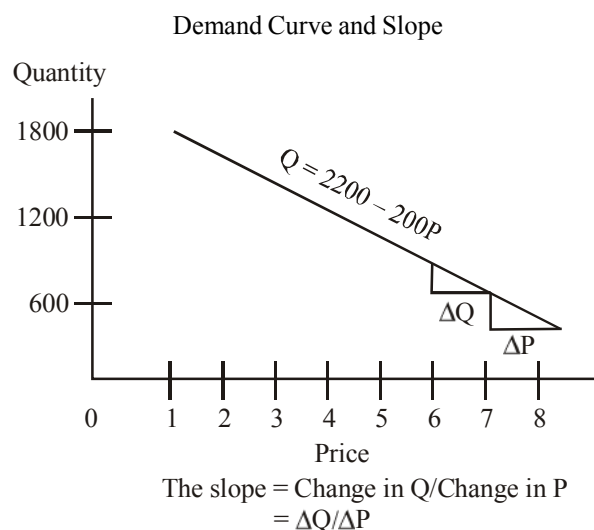
The regression equation is:  $Q = 2200 - 200P$ . The negative slope indicates that quantity demanded is indirectly proportional with price, i.e., as P rises, Q falls and vice versa.

Now that we know:  $Q = 2200 - 200P$ , differentiating Q with respect to P, we get:

$$dQ/dP = -200$$

This is the same as regression coefficient we got earlier.

Graph below depicts the demand curve. The curve is downward sloping, indicating that its slope is negative.

**Linear Supply Function and Slope thereof**

Let the supply of goods at different prices be as per the schedule given:

Unit Price	P	1	2	3	4	5
Quantity	Q	600	800	1000	1200	1400

**NOTES**

We can show that the slope is positive and is equal to 200. The equation can be worked out as before.

**NOTES**

P – Mean of P = p	-2	-1	0	1	2	0
Q – Mean of Q = q	-400	-200	0	200	400	0
pq	800	200	0	200	800	2000
p <sup>2</sup>	4	1	0	1	4	10

Let the equation be:  $Q = a + bp$

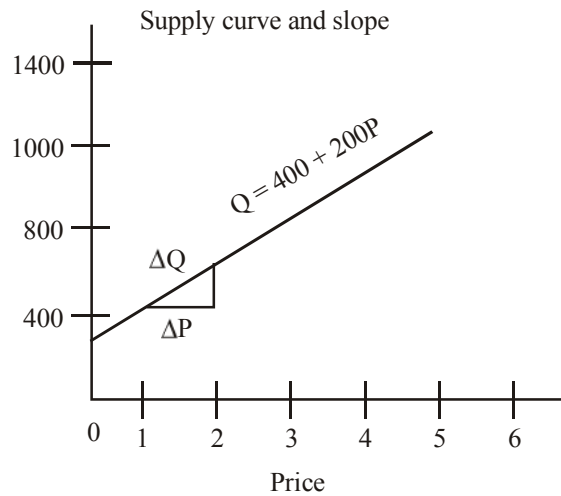
$b = \text{Slope} = \Sigma pq / \Sigma p^2 = 2000 / 10 = 200$

$a = \text{Mean of Q} - b (\text{Mean of P}) = 1000 - 200(3) = 400$

The regression equation:  $Q = 400 + 200p$

Graph below gives the curve.

The slope =  $\Delta Q / \Delta P = 200$



**Slope in the Case of Non-Linear Functions**

Slope in the case of non-linear functions is not constant for all points on the curve, unlike the case with linear functions with same slope for all points on the curve.

Let the cost curve and revenue curve be  $500 + 13Q + 2Q^2$  and  $125Q - 2Q^2$ , respectively.

The cost curve can be derived as follows:

Q	0	5	10	15	20
TC	500	615	830	1145	1560

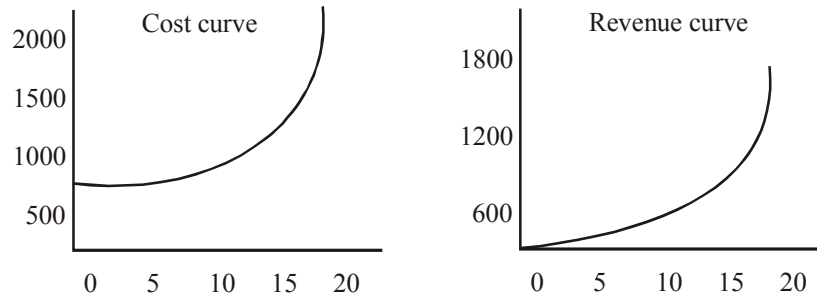


The revenue curve can be derived as follows:

Q	0	5	10	15	20
TR	0	575	1050	1425	1700

**NOTES**

Graphs below depict the cost and revenue curves:



The slope values of the curves are not same at all points. For different magnitude of 'Q', different slope levels exist.

The first derivative of cost function with respect to Q gives the slope. It is:  $13 + 4Q$ . So, slope for different 'Q' values are as tabled below, obtained by putting the value of Q in the above slope function:

Q	0	5	10	15	20
Slope ( $13 + 4Q$ )	13	33	53	73	93

The first derivative of revenue function with respect to Q gives its slope. It is:  $125 + 4Q$ . So, slope for different 'Q' values are as tabled below:

Q:	0	5	10	15	20	
Slope:	125	105	85	65	45	( $125 + 4Q$ )

**Example 3.18:** What is the slope of a line with inclination  $45^\circ$  which passes through the origin?

Take a point P on the line. Draw  $PM \perp OX$ . Then the slope is given by

$$\tan 45^\circ = \frac{MP}{OM} = 1 \text{ by geometry since } MP = OM.$$

**Linear Function with Plural Independent Variables**

So far we dealt with linear functions involving only one independent variable. We can have linear functions with more than one independent variable.

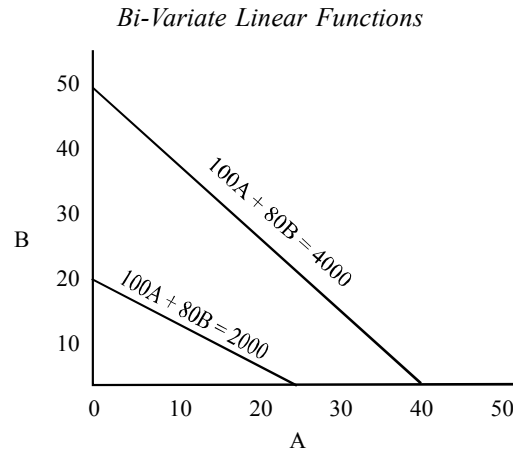
Suppose a firm produces two products A and B. A's contribution per unit is ₹ 100 and B's contribution per unit is ₹ 80. Assume 'a' units of A and 'b' units of B are produced. The total contribution function is a linear function as follows:

$100a + 80b$ . The contribution function can be expressed in a graph in the form of combinations of A and B, giving a particular level of contribution.

Suppose ₹ 2000 and ₹ 4000 levels of contribution are needed. We can plot in the graph combinations of A and B giving ₹ 2000 and ₹ 4000 levels of contribution.

## NOTES

₹ 4000 contribution can be got through 40 units of A or through 50 units of B or combinations of A and B.



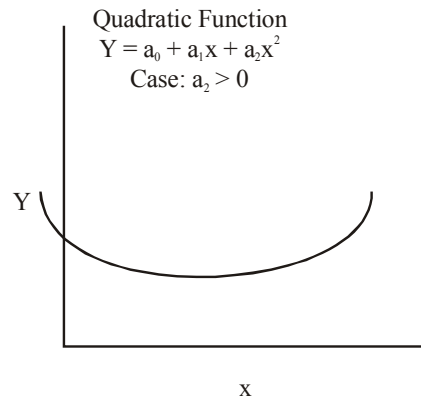
In the graph above, 'A' is taken on x-axis and 'B' is taken on y-axis. The line joining 40 units of A and 50 units of B, represents all combinations of A and B yielding a total contribution of ₹ 4000. Similarly the line joining 20 units of A and 25 units of B represents all combinations of A and B yielding a total contribution of ₹ 2000. All lines parallel to these lines represent combinations of A and B giving certain levels of total contribution. For higher total contribution, the curve shall be farther from origin and vice versa.

If 3 independent variables are involved, we can still have graphical version, a 3 + 1 dimensional graph. Beyond 3 independent variables, the equation form is the only method of presentation.

Suppose there are 'n' independent variables. The general order of linear equation here is:  $Y = a_0 + a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n$ , where  $x_1, \dots, x_n$  are independent variables and  $a_0, a_1, \dots, a_n$  are constants or coefficients.

### Non-Linear Functions

Non-linear functions are simply the 2nd or further higher degree polynomials. We know that the 2nd degree polynomial is a quadratic function, the 3rd degree polynomial is a cubic function and so on. In the graph of quadratic functions if  $a_2 < 0$  then the graph appears like a 'hill'. If  $a_2 > 0$ , then the graph will form a 'Valley' as shown in graph below.



**NOTES**

**Homogeneous Functions**

A homogeneous functions is a special type of function frequently used in economics.

A function is homogeneous of degree  $n$  if when each of its variables is replaced by  $k$  times the variable, the new function is  $kn$  times the original function.

If  $z = f(x, y)$  is homogeneous of degree  $n$ ,

then  $f(kx, ky) = k^n z = k^n f(x, y)$

**Example 3.19:**  $f(x, y) = x^2 - 3xy + 5y^2$  is a homogeneous function of degree 2 because

$$\begin{aligned} f(kx, ky) &= k^2x^2 - 3kx - ky + 5k^2y^2 \\ &= k^2(x^2 - 3xy + 5y^2) = k^2f(x, y) \end{aligned}$$

**Cubic Function**

Let us continue with the quadratic function of  $Q$  = quantity demanded as a function of  $P$  = price, as:  $Q = P^2 - 7P + 10$ . From this we can get the total revenue function,  $TR = PQ = P(P^2 - 7P + 10) = P^3 - 7P^2 + 10P$ . This is a cubic function, a non-linear form of 3rd order polynomial.

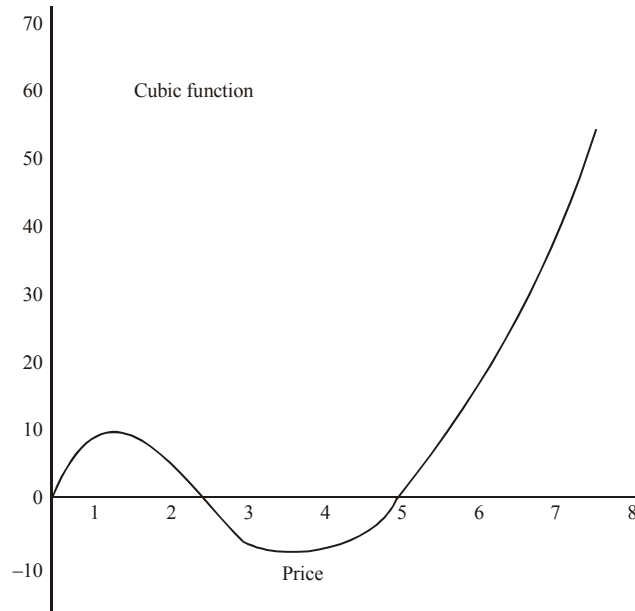
We can plot the curve by assigning values to 'P' and getting those of TR.

The same is tabled below:

Price per unit = P = 0	1	2	3	4	5	6
7						
Total revenue = TR = 0	4	0	-6	-8	0	24
70						
$(P^3 - 7P^2 + 10P)$						

**NOTES**

Graph below presents the revenue function.



The first derivative of the total revenue function is the marginal revenue function. That is, the rate of change in total revenue for a given change in price is  $= 3P^2 - 14P$ . This is the slope of the curve. It varies from place to place on the curve.

**Cobb Douglas Production Function**

If  $L$  stands for labour,  $K$  for capital and  $\alpha, \beta$  are constants such that  $\alpha + \beta = 1$  then in the Cobb Douglas production function, replacing  $L, K$  by  $pL, pK$

$$Y = aL^\alpha K^\beta$$

$$\begin{aligned} \text{We have } a. (pL)^\alpha . (pK)^\beta &= a. p^\alpha L^\alpha . p^\beta K^\beta \\ &= a. p^{\alpha+\beta} L^\alpha K^\beta = p. aL^\alpha K^\beta = pY \end{aligned}$$

since  $\alpha + \beta = 1$

Hence the Cobb Douglas production function is *linearly homogeneous*.

**Example 3.20:** Show that (i)  $Y = \left( \frac{a}{L^r} + \frac{\beta}{K^r} \right)^{-1/r}$

$$(ii) Y = a[\delta K^{-r} + (1 - \delta)L^{-r}]^{-1/r}$$

are homogeneous of degree 1.

**Returns to Scale**

Let  $Y = f(K, L)$  be a linearly homogeneous production function in labour  $L$  and capital  $K$ .

Raising all inputs of  $L, K$   $c$ -fold will raise the output  $c$ -fold.

Since,  $f(cK, cL) = cY$

In other words, a linearly homogeneous production function implies constant returns to scale in production. If no input is used there is no output. If inputs are doubled, output is doubled.

### Demand Functions and Supply Functions

Let  $x$  denotes the quantity of a commodity demanded and  $p$  its price,  $x$  and  $p$  being variables we may write the demand function

$$x = f(p) \text{ showing dependence of } x \text{ on } p \text{ or}$$

$$p = F(x) \text{ showing dependence of } p \text{ on } x$$

These are the explicit forms of the implicit demand function  $g(x, p) = 0$ .

The variables, in the case of a demand function, as in the case of other functions in economics, are hypothetical quantities and not actual observable quantities. Changes in the values of parameters cause shifts in the demand curve.

The arguments given above apply to a supply function also if  $x$  stands for the variable supply. The slope of a supply curve is positive and that of a demand curve is negative, in general.

### Cost Functions and Production Functions

If  $x$  is the quantity produced by a firm at total cost  $C$ , we have the total cost function  $C = f(x)$  explicitly. We may write this in the implicit form:

$$g(C, x) = 0$$

Average cost of production or cost per unit is obtained by dividing total cost by the quantity produced.

$$AC = C/x \text{ or } C = AC.x$$

Cost curves can be obtained from a knowledge of *production functions*. A production relationship is described by inputs associated with specified amounts of outputs. In the production function

$$Y = f(L, K)$$

$L, K$  are quantities of labour and capital respectively required to produce  $Y$ .

If  $P_L$  is the wage rate or price of labour,  $P_K$  the price per unit of capital then we can write the total cost as the sum of all inputs (i.e.,  $L, K$  in this case) times their respective prices:

$$C = LP_L + KP_K$$

### Profit Function

$E = \text{Profit} = \text{Total Revenue} - \text{Total Cost}$ . For the example dealt above, Profit =  $125Q - 2Q^2 - (500 + 13Q + 2Q^2) = 125Q - 2Q^2 - 500 - 13Q - 2Q^2 = -4Q^2 + 112Q - 500$ . Profit for 0, 5, 10, 15 and 20 units are: -500, 40, 220, 280 and 140 respectively.

### NOTES

The slope of the profit curve, i.e., first order derivative with respect to  $e = dE/dQ$  of  $-4Q^2 + 112Q - 500 = -8Q + 112$ . At  $Q = 0, 5, 10, 15$  and  $20$  the slope is:  $112, 72, 32, -8$  and  $-48$ . Graphs below gives the total profit and marginal profit curves.

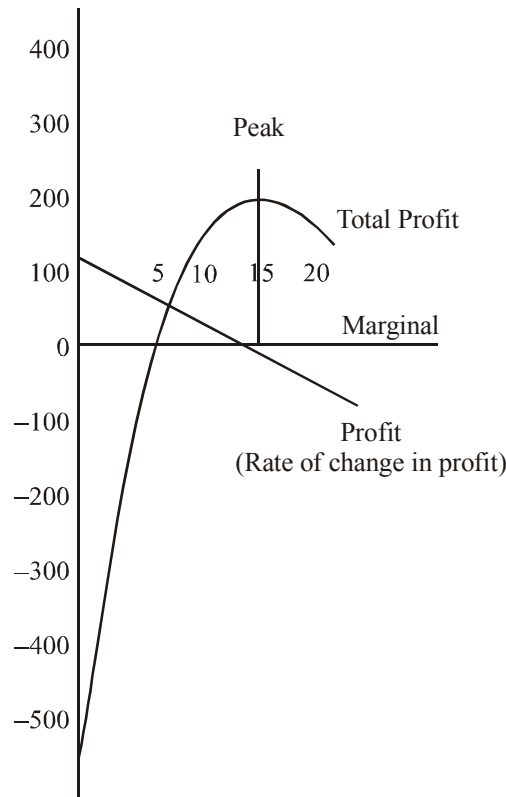
**NOTES**

**What is the profit optimizing output level?**

The profit function is:  $E = -4Q^2 + 112Q - 500$ .

First derivative  $\delta E/\delta Q = -8Q + 112$ . Profit maximizing output is given by letting the first derivative equal to zero and solve for  $Q$ . So,  $-8Q + 112 = 0$ ;  $8Q = 112$  or  $Q = 14$ . At 14 units, profit = 284. To ensure the correctness of the answer, the 2nd derivative must be done and if its value is negative, the answer is correct. And  $d^2E/dQ^2 = -8$ . So our answer is correct.

When marginal profit (i.e., slope of total profit) reaches zero, total profit is maximum. The quantity at this level of profit is profit maximizing quantity. From the following graph we read it as 14 units, the same as we got earlier algebraically.



**3.3 MARGINAL COSTING AND ELASTICITY OF DEMANDS**

There are mainly two techniques of product costing and income determination: (i) Absorption costing; (ii) Marginal costing.

## Absorption Costing

**Absorption costing** is a total cost technique under which total cost (i.e., fixed cost as well as variable cost) is charged as production cost. In other words, in absorption costing, all manufacturing costs are *absorbed* in the cost of the products produced. In this system, fixed factory overheads are absorbed on the basis of a predetermined overhead rate, based on normal capacity. Under/over absorbed overheads are adjusted before computing profit for a particular period. Closing stock is also valued at total cost which includes all direct costs and fixed factory overheads (and sometimes administration overheads also).

Absorption costing is a traditional approach and is also known as *conventional costing* or *full costing*.

## Marginal Costing

An alternative to absorption costing is marginal costing, also known as *variable costing* or *direct costing*. Under this technique, only variable costs are charged as product costs and included in inventory valuation. Fixed manufacturing costs are not allotted to products but are considered as period costs and thus charged directly to Profit and Loss Account of that year. Fixed costs also do not enter in stock valuation.

Both absorption costing and marginal costing treat non-manufacturing costs (i.e., administration, selling and distribution overheads) as period costs. In other words, these are not inventoriable costs.

## Product Costs and Period Costs

Product costs are those costs which become a part of production cost. Such costs are also included in inventory valuation. Period costs, on the other hand, are those costs which are not associated with production. Such costs are treated as an expense of the period in which these are incurred. These do not form part of the cost of products or inventory. These are directly transferred to Profit and Loss Account of the period.

## Meaning of Marginal Cost

Marginal cost is the additional cost of producing an additional unit of product. It is the total of all variable costs. It is composed of all direct costs and variable overheads. The CIMA London has defined marginal cost '*as the amount at any given volume of output by which aggregate costs are changed, if volume of output is increased or decreased by one unit*'. It is the cost of one unit of product which would be avoided if that unit were not produced. An important point is that marginal cost per unit remains unchanged, irrespective of the level of activity.

**Example 3.21:** A company manufactures 100 units of a product per month. Total fixed cost per month is ₹ 5,000 and marginal cost per unit is ₹ 250. The total cost per month will be:

## NOTES

	₹
Marginal (variable) cost of 100 units	25,000
Fixed cost	5,000
	Total cost 30,000

**NOTES**

If output is increased by one unit, the cost will appear as follows:

Marginal cost (101 × 250)	25,250
Fixed cost	5,000
	Total cost 30,250

Thus, the additional cost of producing one additional unit is ₹ 250, which is its marginal cost. However, fixed costs may also increase with the increase in the volume of output, but this may be the result of increase in production capacity.

**Elasticity of Demand through Functions**

Elasticity is a concept very much used in business and economics. Elasticity of demand, supply, etc., are common parlance terms in business studies.

Elasticity of demand to price of the product, to increase in the income of the customer, to price of competitor product (known as cross elasticity of demand), to advertisement and promotion campaigns, etc., are very useful concepts in businesses. All these aim to measure the rate of change of quantity demanded to a given change in price, income, price of competitor product or advertisement and proportional spending.

**Price Elasticity of Demand**

Let us now consider price elasticity of demand. Let it be denoted by 'E'. By definition, E is given by: [Rate of change in quantity demanded/Rate of change in Price]

$$[-\Delta Q/Q]/[\Delta P/P]$$

where Q is the quantity, P is the price, and  $\Delta Q$  and  $\Delta P$  are the changes in Q and P respectively. [Note that the negative sign is used as elasticity is both directional and dimensional and as Q (quantity) varies opposite to the P (price), the '-' sign is attenuated to the  $\Delta Q$ .]

$$\begin{aligned} E &= - [\Delta Q/Q] \times [P/\Delta P] \\ &= - [P/Q] \times [\Delta Q/\Delta P] \end{aligned}$$

You must remember  $\Delta Q/\Delta P$  is a slope function and is slope of quantity with respect to price. So, elasticity of demand is slope of demand curve multiplied by  $- [P/Q]$ . For all normal goods, and luxury goods the price elasticity is negative, but conventionally the -ve sign is ignored.

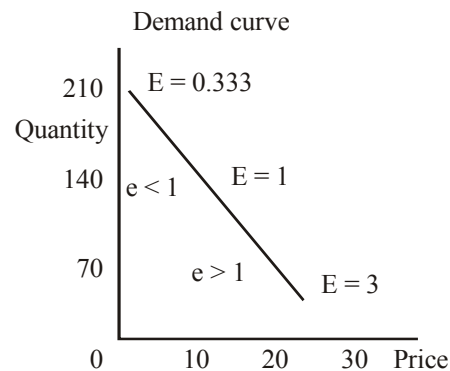
Let the demand function be  $Q = 280 - 7P$ ; its elasticity (E) is given by  $-(P/Q) \times \text{Slope}$ . The slope is simply the first derivative of Q with respect to P. So,  $dQ/$



$dP = -7$ . The E for the different values of P is computed and given in the table and the graph gives the demand curve.

When elasticity is  $>1$ , the demand is said to be price elastic and reducing price, more volume can be sold. When elasticity is  $<1$ , the demand is inelastic. By reducing price you cannot sell more; at the same time by raising the price, your sales volume is not going to be severely pruned.

## NOTES



P	10	20	30
Corresponding Q	210	140	70
Slope	-7	-7	-7
Elasticity = $-(P/Q) \times (\text{Slope})$	0.333	1	3
Total Revenue (P $\times$ Q)	2100	2800	2100

When price is reduced from ₹ 30 to ₹ 20, i.e., by 33%, then Q, i.e., the sales volume has increased by 100% from 70 to 140. That is, when elasticity was high, a reduction in P enabled the firm to enhance sales. Also, the sales revenue increased from ₹ 2100 to ₹ 2800. When price elasticity was less than 1, an increase in price will get good revenue. See, when price was increased from ₹ 10 to ₹ 20 by 100% volume of sales declined only by 33% from 210 to 140. At the same time, revenue increased from ₹ 2100 to ₹ 2800. Thus for the seller, it pays to rise prices when he has price inelastic demand curve. Similarly, when he forces an elastic demand curve, it pays to reduce price.

The price elasticity is  $>1$  beyond the price level of ₹ 20. So, any price rise from ₹ 20 will have the seller with low sales. But price reduction to ₹ 20 from higher price level will benefit him. The price elasticity is  $<1$ , below the price level of ₹ 20. So, any price reduction from ₹ 20 will not benefit the seller, but any price rise, movement in price towards the unit elasticity point from any previous position of elasticity pays off the seller well.

You can notice the use of slope in computing price elasticity. Price elasticity is simply slope times  $P/Q$ . You can also note, given the slope, price elasticity is directly proportional to price and inversely proportional to quantity.

**Income Elasticity of Demand**

Income elasticity of demand process is defined as the ratio of rate of change in quantity demanded to rate of change in income of the consumers.

**NOTES**

Income elasticity =  $[\Delta Q/Q] / [\Delta I/I]$ , where Q is the quantity, I is the income, and  $\Delta Q$  and 'I' are the changes in Q and I. [Note that the negative sign is used as elasticity in both directional and dimensional, and as Q (quantity) varies opposite to the price.

$$= [\Delta Q/Q] \times [I/\Delta I]$$

$$= -[I/Q] \times [\Delta Q/\Delta I]$$

Suppose the following demand income function is present:  $Q = 500 + 0.05I$ , where Q is quantity demanded and I is income. Then the following table of Q and I ordered pairs can be worked out:

I	1000	1500	2000	2500
Q	550	975	600	625
Slope: $\Delta Q/\Delta I$	0.05	0.05	0.05	0.05
Elasticity = $(I/Q) \times (\Delta Q/\Delta I)$	0.091	0.130	0.167	0.2

[Slope of the income-demand function is the 1st order derivative of that function. We know the function is :  $Q = 500 + 0.05I$ . Its 1st order derivative:  $\delta Q/\delta I = 0.05$ .

**Cross Elasticity of Demand**

It is the ratio of rate of change in quantity demanded of a product to the rate of change in price of a related product.

If X any Y are complementary goods [that is, use of one needs use of the other too, like car and petrol], then as the price of Y rises, demand for X falls and vice versa.

Cross elasticity (Complementary goods)  $Q_x$  and  $P_y = [-\Delta Q_x/Q_x] / [\Delta P_y/P_y]$ , where  $Q_x$  is the quantity of petrol driven car,  $P_y$  is the price of Y (petrol) and  $Q_x$  and  $\Delta P_y$  are the changes in quantity and price respectively. [Note that the negative sign is used as elasticity in both directional and dimensional and as Q (quantity) varies opposite to the price.

$$= [-\Delta Q_x/Q_x] \times [P_y/\Delta P_y]$$

$$= -[P_y/Q_x] \times [\Delta Q_x/\Delta P_y]$$

If X and Y are competitive goods [that is use of one results in non-use of the other, like coffee and tea), then as the price of Y rises, demand for X rises and vice versa.

$$\text{Cross elasticity (substitute goods) } Q_x \text{ and } P_y = [\Delta Q_x/Q_x] / [\Delta P_y/P_y]$$

$$= [\Delta Q_x/Q_x] \times [P_y/\Delta P_y]$$

$$= [P_y/Q_x] \times [\Delta Q_x/\Delta P_y]$$

### Advertisement and Promotion Elasticity of Demand

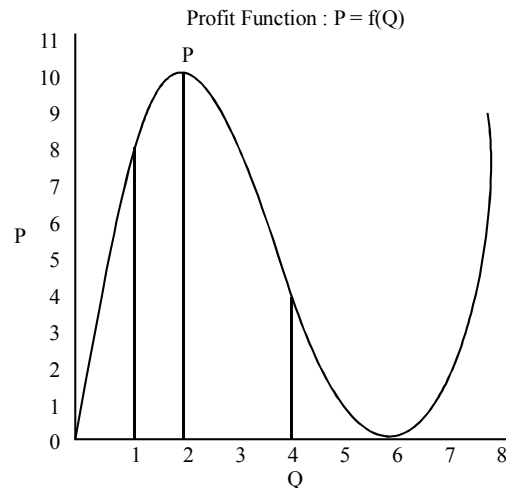
It is defined as the ratio of rate of change in quantity demanded to rate of change in amount spent on advertisement and promotion.

Advertisement and Promotion Elasticity of demand =  $[(\Delta/Q) \times (\Delta Q/\Delta A)]$  where A is the amount spent on advertisement. Here the slope is normally positive and demand is elastic to advertisement and promotion. That is why big companies spend lot on advertisement promotion.

#### 3.3.1 Maxima and Minima of a Function

A function, especially a higher degree function, might be highest for some value of the independent variable and lowest for some value of the independent variable and there may be many 'highs' and 'lows'. These 'highs' are called 'maxima' and the 'lows' are called the 'minima'. Hence the need to study the maxima and minima of functions.

Look at the graph given below for the function  $P = f(Q)$



Consider the graph in the interval  $Q = 1$  to  $4$ . The  $f(Q)$  is increasing to the left of  $Q = 2$  and decreasing to the right of  $Q = 2$  and is highest at  $P$ , when  $Q = 2$ . Consider the interval,  $Q = 5$  to  $7$ . To the left of  $Q = 6$ ,  $f(Q)$  is decreasing and to the right of  $Q = 6$ ,  $f(Q)$  is increasing and is lowest when  $Q = 6$ . When  $f(Q)$  is rising, the slope is positive and when decreasing, the slope is negative. And, when slope is zero, either  $f(x)$  is maximum or minimum. So, if the slope is zero, how can we say definitely that  $f(x)$  is at its maximum or at its minimum? Here comes to our rescue the 2nd order derivative or  $d^2P/dQ^2$ . If the value of  $d^2P/dQ^2 < 0$ , i.e., negative,  $f(Q)$  is at its maximum point and if  $d^2P/dQ^2 > 0$ , i.e., positive,  $f(Q)$  is at its minimum point.

#### Example 3.27:

Suppose the profit function is,  $P = Q^3/3 - 4Q^2 + 12Q$ .

Find the  $Q$  for which profit is maximum and the  $Q$  for which profit is minimum.

### NOTES

**NOTES****Solution:**

$$\text{Profit} = Q^3/3 - 4Q^2 + 12Q;$$

$$\text{1st Derivative: } dP/dQ = Q^2 - 8Q + 12.$$

If we set this as equal to 0, we get:  $Q^2 - 8Q + 12 = 0$ .

$$\text{So, } (Q-2)(Q-6) = 0.$$

$$\text{So, } (Q-2) = 0 \text{ or } (Q-6) = 0.$$

$$\text{So, } (Q) = 2 \text{ or } Q = 6.$$

$$\text{2nd Derivative: } d^2P/dQ^2 \text{ of } Q^2 - 8Q + 12 = 2Q - 8;$$

If we put the value,  $Q = 2$ , in the 2nd derivative, the value of 2nd derivative becomes:  $2(2) - 8 = -4$  and if we put  $Q = 6$ , its value is:  $2(6) - 8 = 4$ .

Elementary calculus says, if the value of 2nd derivative is negative, the profit is maximum at that level of output. Here, for  $Q = 2$ , the 2nd derivative is negative and that at  $Q = 2$ , profit is highest.

We can calculate it. Take the profit equation:  $\text{Profit} = Q^3/3 - 4Q^2 + 12Q$ . Put  $Q = 2$  here. We get the profit as  $\text{₹ } 8/3 - 4(4) + 12(2) = \text{₹ } 10.67$ . At  $Q = 6$ , we get the Profit as:  $216/3 - (4 \times 26) + 12(6) = 72 - 144 - 72 = 0$ .

We can get ordered pair points for profit for the different Q levels as below:

Q	0	1	2	3	4	5	6	7
Profit = P	0	9	10.67	9	5.33	1.67	0	2.33

These points are plotted in the graph of the profit function plotted above.

**Note:** A function can have many maxima and corresponding minima.

**Check Your Progress**

- Fill in the blanks with appropriate words.
  - The programming technique of \_\_\_\_\_ mode describes the technology and the economics of a business through simultaneous equations and inequalities.
  - The statistical technique of \_\_\_\_\_ examines the past trends of relationships between variables such as sales volume.
  - The \_\_\_\_\_ model reflects situations where the outcomes of the decisions are not unique and cannot be predicted with total certainty.
  - A function expressed in terms of polynomials and roots of polynomials is an \_\_\_\_\_ function.
- State whether the following statements are true or false.
  - Extrapolation means reading a value which lies between two extreme points.

- b. Acceptance sampling is that phase of statistical quality control which attempts to decide whether to reject or accept a lot having the desired quantity level.
- c. Simulation results are inferential in nature and may not be very precise.
- d. If  $f'(x) > 0$ , then  $f(x)$  is increasing function of  $x$  and if  $f'(x) < 0$ , then  $f(x)$  is decreasing function of  $x$ .

**NOTES**

### 3.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

- |                    |               |
|--------------------|---------------|
| 1. a. Mathematical | b. Regression |
| c. Probabilistic   | d. Algebraic  |
| 2. a. False        | b. True       |
| c. True            | d. True       |

### 3.5 SUMMARY

- The way in which one variable depends on other variables is described by means of functions.
- The point  $(c, f(c))$  is called a maximum point of  $y = f(x)$ , if (i)  $f(c + h) \leq f(c)$ , and (ii)  $f(c - h) \leq f(c)$  for small  $h \geq 0$ .  $f(c)$  itself is called a maximum value of  $f(x)$ .
- A function  $f(x)$  is said to be increasing (decreasing) if  $f(x + c) \geq f(x) \geq f(x - c)$  [ $f(x + c) \leq f(x) \leq f(x - c)$ ] for all  $c \geq 0$ .

### 3.6 KEY WORDS

- **Algebraic functions:** Algebraic functions are obtained through a finite number of algebraic operations like addition, subtraction, multiplication, division and through solving a finite number of algebraic equations. Polynomials, rational and radical functions are algebraic
- **Absorption costing:** It is a total cost technique under which total cost (i.e., fixed cost as well as variable cost) is charged as production cost
- **Maxima and Minima:** Maxima and minima are the largest value (maximum) or the smallest value (minimum), that a function takes in a point either within a given neighbourhood or on the function domain in its entirety.

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### 3.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

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**NOTES****Short Answer Questions**

1. Differentiate between even and odd functions.
2. What is a real-valued function?

**Long Answer Questions**

1. Discuss the types of variables and functions with examples.
2. Discuss the maxima and minima of the following functions.

(i)  $x^5 - 5x^4 + 5x^3 - 10$

(ii)  $x^3 - 3x^2 - 9$

(iii)  $x + \frac{1}{x+1}$

(iv)  $\frac{(x+3)^2}{x^2+1}$

(v)  $x + \sin 2x$  (for  $0 \leq x \leq \pi$ )

(vi)  $\cos x + \cos 3x$  (for  $0 \leq x \leq \pi$ )

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### 3.8 FURTHER READINGS

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# UNIT 4 LINEAR PROGRAMMING: AN INTRODUCTION

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## NOTES

### Structure

- 4.0 Introduction
- 4.1 Objectives
- 4.2 Formulation of Different Types of Linear Programming
  - 4.2.1 Meaning of Linear Programming
  - 4.2.2 Importance and Practical Implementation in Industry
  - 4.2.3 Components of a Linear Programming Problem
  - 4.2.4 Formulation of a Linear Programming Problem
- 4.3 Canonical and Standard Forms of LPP
- 4.4 Answers to Check Your Progress Questions
- 4.5 Summary
- 4.6 Key Words
- 4.7 Self Assessment Questions and Exercises
- 4.8 Further Readings

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## 4.0 INTRODUCTION

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This unit introduces you to linear programming, where you will first learn the various components of an LP problem. You will also learn how to formulate an LP problem and about the standard and canonical forms of LPP.

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## 4.1 OBJECTIVES

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After going through this unit, you will be able to:

- Understand the meaning of linear programming
- Analyse the various fields where LP can be applied
- Formulate an LP problem
- Describe the standard form of LP problems

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## 4.2 FORMULATION OF DIFFERENT TYPES OF LINEAR PROGRAMMING

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Decision-making has always been very important in the business and industrial world, particularly with regard to the problems concerning production of commodities, such as, which commodity/commodities should be produced and in what quantities and also, by which process or processes should they be produced are some of the main questions before a production manager. Alfred Marshall, in

## NOTES

this connection points out that the businessman always studies his production function and input prices and substitutes one input for another till his costs become the minimum possible. This sort of substitution, in the opinion of Marshall, is done by the businessman's trained instinct rather than with formal calculations. But now there does exist a method of formal calculations often termed as Linear Programming. This method was first formulated by a Russian mathematician L.V. Kantorovich, but it was developed later in 1947 by George B. Dantzig 'for the purpose of scheduling the complicated procurement activities of the United States Air Force'. Today, this method is being used in solving a wide range of practical business problems. The advent of electronic computers has further increased its applications to solve many other problems in industry. It is being considered as one of the most versatile management techniques.

### **4.2.1 Meaning of Linear Programming**

Linear Programming (LP) is a major innovation since World War II in the field of business decision-making, particularly under conditions of certainty. The word 'Linear' means that the relationships are represented by straight lines, i.e., the relationships are of the form  $y = a + bx$  and the word 'Programming' means taking decisions systematically. Thus, LP is a decision-making technique under given constraints on the assumption that the relationships amongst the variables representing different phenomena happen to be linear. In fact, Dantzig originally called it 'programming of interdependent activities in a linear structure', but later on shortened it to 'Linear Programming'. LP is generally used in solving maximization (sales or profit maximization) or minimization (cost minimization) problems subject to certain assumptions. Putting in a formal way, 'Linear Programming is the maximization (or minimization) of a linear function of variables subject to a constraint of linear inequalities.' Hence, LP is a mathematical technique designed to assist the organization in optimally allocating its available resources under conditions of certainty in problems of scheduling, product-mix and so on.

### **4.2.2 Importance and Practical Implementation in Industry**

The problem for which LP provides a solution may be stated as, maximize or minimize for some dependent variable which is a function of several independent variables when the independent variables are subject to various restrictions. The dependent variable is usually some economic objective such as profits, production, costs, workweeks, tonnage to be shipped and so on. More profits are generally preferred to less profits and lower costs are preferred to higher costs. Hence, it is appropriate to represent either maximization or minimization of the dependent variable as one of the firm's objective. LP is usually concerned with such objectives under given constraints with linearity assumptions. In fact, it is powerful to take in its stride a wide range of business applications. The applications of LP are numerous and are increasing everyday. LP is extensively used in solving resource allocation problems. Production planning and scheduling, transportation, sales and advertising,



financial planning, portfolio analysis, corporate planning, etc., are some of its most fertile application areas. More specifically, LP has been successfully applied in the following fields:

- (i) **Agriculture:** LP can be applied in farm management problems as it relates to the allocation of resources such as acreage, labour, water supply or working capital in such a way that it maximizes net revenue.
- (ii) **Contract Awards:** Evaluation of tenders by recourse to LP guarantees that the awards are made in the cheapest way.
- (iii) **Industries:** Applications of LP in business and industry are of the most diverse type. Transportation problems concerning cost minimization can be solved by this technique. The technique can also be adopted in solving problems of production (product-mix) and inventory control.

Thus, LP is the most widely used technique of decision-making in business and industry in modern times in various fields.

#### 4.2.3 Components of a Linear Programming Problem

There are certain basic concepts and notations to be first understood for easy adoption of the LP technique. A brief mention of such concepts is as follows:

- (i) **Linearity.** The term linearity implies straight line or proportional relationships among the relevant variables. Linearity in economic theory is known as constant returns, which means that if the amount of the input doubles, the corresponding outputs and profits are also doubled. Linearity assumption, thus, implies that two machines and two workers can produce twice as much as one machine and one worker; four machines and four workers twice as much as two machines and two workers, and so on.
- (ii) **Process and its level.** Process means the combination of particular inputs to produce a particular output. In a process, factors of production are used in fixed ratios, of course, depending upon technology and as such no substitution is possible with a process. There may be many processes open to a firm for producing a commodity and one process can be substituted for another. There is, thus, no interference of one process with another when two or more processes are used simultaneously. If a product can be produced in two different ways, then there are two different processes (or activities or decision variables) for the purpose of a linear programme.
- (iii) **Criterion Function.** This is also known as the objective function which states whether the determinants of the quantity should be maximized or minimized. For example, revenue or profit is such a function when it is to be maximized or cost is such a function when the problem is to minimize it. An objective function should include all the possible activities with the revenue (profit) or cost coefficients per unit of production or acquisition. The goal may be either to maximize this function or to minimize this function. In

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symbolic form, let  $ZX$  denote the value of the objective function at the  $X$  level of the activities included in it. This is the total sum of individual activities produced at a specified level. The activities are denoted as  $j = 1, 2, \dots, n$ . The revenue or cost coefficient of the  $j$ th activity is represented by  $C_j$ . Thus,  $2X_1$  implies that  $X$  units of activity  $j = 1$  yields a profit (or loss) of  $C_1 = 2$ .

- (iv) **Constraints or Inequalities.** These are the limitations under which one has to plan and decide, i.e., restrictions imposed upon decision variables. For example, a certain machine requires one worker to operate; another machine requires at least four workers (i.e.,  $> 4$ ); there are at most 20 machine hours (i.e.,  $< 20$ ) available; the weight of the product should be, say 10 lbs and so on, are all examples of constraints or what are known as inequalities. Inequalities like  $X > C$  (reads  $X$  is greater than  $C$  or  $X < C$  (reads  $X$  is less than  $C$ ) are termed as strict inequalities. The constraints may be in the form of weak inequalities like  $X \leq C$  (reads  $X$  is less than or equal to  $C$ ) or  $X \geq C$  (reads  $X$  is greater than or equal to  $C$ ). Constraints may be in the form of strict equalities like  $X = C$  (reads  $X$  is equal to  $C$ ).

Let  $b_i$  denote the quantity  $b$  of resource  $i$  available for use in various production processes. The coefficient attached to resource  $i$  is the quantity of resource  $i$  required for the production of one unit of product  $j$ .

- (v) **Feasible Solutions.** These are all those possible solutions which can be worked upon under given constraints. The region comprising all feasible solutions is referred to as *Feasible Region*.
- (vi) **Optimum Solution.** Optimum solution is the best of all the feasible solutions.

### General form of the Linear Programming Model

Linear Programming problem mathematically can be stated as follows:

Choose the quantities,

$$X_j \geq 0 \quad (j = 1, \dots, n) \quad \dots(4.1)$$

This is also known as the non-negativity condition and in simple terms means that no  $X$  can be negative.

To maximize,

$$Z = \sum_{j=1}^n C_j X_j \quad \dots(4.2)$$

Subject to the constraints,

$$\sum_{j=1}^n a_{ij} X_j \leq b_i \quad (i = 1, \dots, m) \quad \dots(4.3)$$

This is the usual structure of a linear programming model in the simplest possible form. This model can be interpreted as a profit maximization situation, where  $n$  production activities are pursued at level  $X_j$  which have to be decided upon, subject to a limited amount of  $m$  resources being available. Each unit of the  $j$ th activity yields a return  $C$  and uses an amount  $a_{ij}$  of the  $i$ th resource.  $Z$  denotes the optimal value of the objective function for a given system.

### Assumptions or conditions to be fulfilled

LP model is based on the assumptions of proportionality, certainty, additivity, continuity and finite choices.

**Proportionality** is assumed in the objective function and the constraint inequalities. In economic terminology, this means that there are constant returns to scale, i.e., if one unit of a product contributes ₹ 5 toward profit, then 2 units will contribute ₹ 10, 4 units ₹ 20 and so on.

**Certainty** assumption means the prior knowledge of all the coefficients in the objective function, the coefficients of the constraints and the resource values. LP model operates only under conditions of certainty.

**Additivity** assumption means that the total of all the activities is given by the sum of each activity conducted separately. For example, the total profit in the objective function is equal to the sum of the profit contributed by each of the products separately.

**Continuity** assumption means that the decision variables are continuous. Accordingly, the combinations of output with fractional values, in case of product-mix problems, are possible and obtained frequently.

**Finite choices** assumption implies that finite number of choices are available to a decision-maker and the decision variables do not assume negative values.

### 4.2.4 Formulation of a Linear Programming Problem

The procedure for mathematical formulation of an LPP consists of the following steps:

**Step 1:** The decision variables of the problem are noted.

**Step 2:** The objective function to be optimized (maximized or minimized) as a linear function of the decision variables is formulated.

**Step 3:** The other conditions of the problem such as resource limitation, market constraints, interrelations between variables and so on, are formulated as linear inequations or equations in terms of the decision variables.

**Step 4:** The non-negativity constraint from the considerations is added so that the negative values of the decision variables do not have any valid physical interpretation.

The objective function, the set of constraints and the non-negative constraint together form a linear programming problem.

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**General formulation of a linear programming problem**

The general formulation of the LPP can be stated as follows:

In order to find the values of  $n$  decision variables  $x_1, x_2, \dots, x_n$  to maximize or minimize the objective function,

$$Z = C_1x_1 + C_2x_2 + \dots + C_nx_n \quad \dots (4.4)$$

$$\left. \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n (\leq, =, \geq) b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n (\leq, =, \geq) b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n (\leq, =, \geq) b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n (\leq, =, \geq) b_m \end{array} \right\} \quad \dots (4.5)$$

Here, the constraints can be inequality  $\leq$  or  $\geq$  or even in the form of an equation ( $=$ ) and finally satisfy the non-negative restrictions:

$$x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \quad \dots (4.6)$$

**Matrix form of a linear programming problem**

The LPP can be expressed in the matrix form as follows:

Maximize or minimize  $Z = Cx \rightarrow$  Objective function

Subject to  $Ax (\leq, =, \geq) b \rightarrow$  Constant equation

$b > 0, x \geq 0 \rightarrow$  Non-negativity restrictions

Where,  $x = (x_1, x_2, \dots, x_n)$

$$C = (C_1, C_2, \dots, C_n)$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

**Example 4.1:** A manufacturer produces two types of models,  $M_1$  and  $M_2$ . Each model of the type  $M_1$  requires 4 hours of grinding and 2 hours of polishing, whereas each model of the type  $M_2$  requires 2 hours of grinding and 5 hours of polishing. The manufacturers have 2 grinders and 3 polishers. Each grinder works 40 hours a week and each polisher works for 60 hours a week. The profit on  $M_1$  model is ₹ 3.00 and on model  $M_2$  is ₹ 4.00. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models, so that he may make the maximum profit in a week?

**Solution:**

**Decision variables:** Let  $X_1$  and  $X_2$  be the number of units of  $M_1$  and  $M_2$ .

**Objective function:** Since the profit on both the models are given, we have to maximize the profit, viz.,

$$\text{Max } Z = 3X_1 + 4X_2$$

**Constraints:** There are two constraints: one for grinding and the other for polishing.

The number of hours available on each grinder for one week is 40 hours. There are 2 grinders. Hence, the manufacturer does not have more than  $2 \times 40 = 80$  hours for grinding.  $M_1$  requires 4 hours of grinding and  $M_2$  requires 2 hours of grinding.

The grinding constraint is given by,

$$4X_1 + 2X_2 \leq 80$$

Since there are 3 polishers, the available time for polishing in a week is given by  $3 \times 60 = 180$ .  $M_1$  requires 2 hours of polishing and  $M_2$  requires 5 hours of polishing. Hence, we have  $2X_1 + 5X_2 \leq 180$

Thus, we have,

$$\text{Max } Z = 3X_1 + 4X_2$$

$$\text{Subject to } 4X_1 + 2X_2 \leq 80$$

$$2X_1 + 5X_2 \leq 180$$

$$X_1, X_2 \geq 0$$

**Example 4.2:** A firm produces two different types of products, Product M and Product N. The firm uses the same machinery for manufacturing both the products. One unit of Product M requires 10 minutes while one unit of Product N requires 2 minutes. The maximum hours the machine can function optimally for a week is 35 hours. The raw material requirement for Product M is 1 kg while that of Product N is 0.5 kg. Also, the market constraint on product M is 600 kg, while that of Product N is 800 units per week. The cost of manufacturing Product M is ₹ 5 per unit and it is sold at ₹ 10; while the cost of Product N is ₹ 6 per unit and sold at ₹ 8 per unit. Calculate the total number of units of Product M and Product N that should be produced per week, so as to derive maximum profit.

**Solution:**

**Decision variables:** Let  $X_1$  and  $X_2$  be the number of products of A and B.

**Objective function:** Cost of product A per unit is ₹ 5 and it is sold at ₹ 10 per unit.

$$\therefore \text{Profit on one unit of product } A = 10 - 5 = 5.$$

$$\therefore X_1 \text{ units of product } A \text{ contribute a profit of } ₹ 5X_1 \text{ from one unit of product.}$$

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Similarly, profit on one unit of  $B = 8 - 6 = 2$ .

$\therefore X_2$  units of product  $B$  contribute a profit of ₹  $2X_2$ .

$\therefore$  The objective function is given by,

$$\text{Max } Z = 5X_1 + 2X_2$$

**Constraints:** Time requirement constraint is given by,

$$10X_1 + 2X_2 \leq (35 \times 60)$$

$$10X_1 + 2X_2 \leq 2100$$

Raw material constraint is given by,

$$X_1 + 0.5X_2 \leq 600$$

Market demand on product  $B$  is 800 units every week.

$$\therefore X_2 \geq 800$$

The complete LPP is,

$$\text{Max } Z = 5X_1 + 2X_2$$

Subject to,  $10X_1 + 2X_2 \leq 2100$

$$X_1 + 0.5X_2 \leq 600$$

$$X_2 \geq 800$$

$$X_1, X_2 \geq 0$$

**Example 4.3:** A person requires 10, 12 and 12 units of chemicals  $A$ ,  $B$  and  $C$  respectively for his garden. A liquid product contains 5, 2 and 1 units of  $A$ ,  $B$  and  $C$  respectively per jar. A dry product contains 1, 2 and 4 units of  $A$ ,  $B$ ,  $C$  per carton. If the liquid product sells for ₹ 3 per jar and the dry product sells for ₹ 2 per carton, what should be the number of jars that needs to be purchased, in order to bring down the cost and meet the requirements?

**Solution:**

**Decision variables:** Let  $X_1$  and  $X_2$  be the number of units of liquid and dry products.

**Objective function:** Since the cost for the products is given, we have to minimize the cost.

$$\text{Min } Z = 3X_1 + 2X_2$$

**Constraints:** As there are three chemicals and their requirements are given, we have three constraints for these three chemicals.

$$5X_1 + X_2 \geq 10$$

$$2X_1 + 2X_2 \geq 12$$

$$X_1 + 4X_2 \geq 12$$

Hence, the complete LPP is,

$$\text{Min } Z = 3X_1 + 2X_2$$

Subject to,

$$5X_1 + X_2 \geq 10$$

$$2X_1 + 2X_2 \geq 12$$

$$X_1 + 4X_2 \geq 12$$

$$X_1, X_2 \geq 0$$

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**Example 4.4:** A paper mill produces two grades of paper,  $X$  and  $Y$ . Because of raw material restrictions, it cannot produce more than 400 tonnes of grade  $X$  and 300 tonnes of grade  $Y$  in a week. There are 160 production hours in a week. It requires 0.2 and 0.4 hours to produce a tonne of products  $X$  and  $Y$ , respectively with corresponding profits of ₹ 200 and ₹ 500 per tonne. Formulate this as an LPP to maximize profit and find the optimum product mix.

**Solution:**

**Decision variables:** Let  $X_1$  and  $X_2$  be the number of units of the two grades of paper,  $X$  and  $Y$ .

**Objective function:** Since the profit for the two grades of paper  $X$  and  $Y$  are given, the objective function is to maximize the profit.

$$\text{Max } Z = 200X_1 + 500X_2$$

**Constraints:** There are two constraints, one with reference to raw material, and the other with reference to production hours.

$$\text{Max } Z = 200X_1 + 500X_2$$

Subject to,

$$X_1 \leq 400$$

$$X_2 \leq 300$$

$$0.2X_1 + 0.4X_2 \leq 160$$

Non-negative restriction  $X_1, X_2 \geq 0$

**Example 4.5:** A company manufactures two products,  $A$  and  $B$ . Each unit of  $B$  takes twice as long to produce as one unit of  $A$  and if the company were to produce only  $A$ , it would have time to produce 2000 units per day. The availability of the raw material is enough to produce 1500 units per day of both  $A$  and  $B$  together. Product  $B$  requiring a special ingredient, only 600 units of it can be made per day. If  $A$  fetches a profit of ₹ 2 per unit and  $B$  a profit of ₹ 4 per unit, find the optimum product mix by the graphical method.

**Solution:** Let  $X_1$  and  $X_2$  be the number of units of products  $A$  and  $B$  respectively.

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The profit after selling these two products is given by the objective function,

$$\text{Max } Z = 2X_1 + 4X_2$$

Since the company can produce at the most 2000 units of the product in a day and Product *B* requires twice as much time as that of Product *A*, production restriction is given by,

$$X_1 + 2X_2 \leq 2000$$

Since the raw material is sufficient to produce 1500 units per day of both *A* and *B*, we have  $X_1 + X_2 \leq 1500$ .

There are special ingredients for Product *B*; hence, we have  $X_2 \leq 600$ .

Also, since the company cannot produce negative quantities  $X_1 \geq 0$  and  $X_2 \geq 0$ .

Hence, the problem can be finally put in the form:

Find  $X_1$  and  $X_2$  such that the profits,  $Z = 2X_1 + 4X_2$  is maximum.

Subject to,  $X_1 + 2X_2 \leq 2000$

$$X_1 + X_2 \leq 1500$$

$$X_2 \leq 600$$

$$X_1, X_2 \geq 0$$

**Example 4.6:** A firm manufactures 3 products *A*, *B* and *C*. The profits are ₹ 3, ₹ 2 and ₹ 4 respectively. The firm has 2 machines and the following is the required processing time in minutes for each machine on each product.

		Product		
		<i>A</i>	<i>B</i>	<i>C</i>
Machines	<i>C</i>	4	3	5
	<i>D</i>	3	2	4

Machines *C* and *D* have 2000 and 2500 machine minutes respectively. The firm must manufacture 100 units of *A*, 200 units of *B* and 50 units of *C*, but not more than 150 units of *A*. Set up an LP problem to maximize the profit.

**Solution:** Let  $X_1, X_2, X_3$  be the number of units of the product *A*, *B*, *C* respectively.

Since the profits are ₹ 3, ₹ 2 and ₹ 4 respectively, the total profit gained by the firm after selling these three products is given by,

$$Z = 3X_1 + 2X_2 + 4X_3$$

The total number of minutes required in producing these three products at Machine *C* is given by  $4X_1 + 3X_2 + 5X_3$  and at Machine *D* is given by  $3X_1 + 2X_2 + 4X_3$ .



The restrictions on machines  $C$  and  $D$  are given by 2000 minutes and 2500 minutes.

$$4X_1 + 3X_2 + 5X_3 \leq 2000$$

$$3X_1 + 2X_2 + 4X_3 \leq 2500$$

Also, since the firm manufactures 100 units of  $A$ , 200 units of  $B$  and 50 units of  $C$ , but not more than 150 units of  $A$ , the further restriction becomes,

$$100 \leq X_1 \leq 150$$

$$200 \leq X_2 \leq 0$$

$$50 \leq X_3 \leq 0$$

Hence, the allocation problem of the firm can be finally put in the form:

Find the value of  $X_1, X_2, X_3$  so as to maximize,

$$Z = 3X_1 + 2X_2 + 4X_3$$

Subject to the constraints,

$$4X_1 + 3X_2 + 5X_3 \leq 2000$$

$$3X_1 + 2X_2 + 4X_3 \leq 2500$$

$$100 \leq X_1 \leq 150, 200 \leq X_2 \leq 0, 50 \leq X_3 \leq 0$$

**Example 4.7:** A peasant has a 100-acre farm. He can sell all potatoes, cabbage or brinjals and can increase the cost to get ₹ 1.00 per kg for potatoes, ₹ 0.75 a head for cabbage and ₹ 2.00 per kg for brinjals. The average yield per acre is 2000 kg of potatoes, 3000 heads of cabbage and 1000 kg of brinjals. Fertilizers can be bought at ₹ 0.50 per kg and the amount needed per acre is 100 kg each for potatoes and cabbage and 50 kg for brinjals. The manpower required for sowing, cultivating and harvesting per acre is 5 man-days for potatoes and brinjals, and 6 man-days for cabbage. A total of 400 man-days of labour is available at ₹ 20 per man-day. Solve this example as a linear programming model to increase the peasant's profit.

**Solution:** Let  $X_1, X_2, X_3$  be the area of his farm to grow potatoes, cabbage and brinjals respectively. The peasant produces  $2000X_1$  kg of potatoes,  $3000X_2$  heads of cabbage and  $1000X_3$  kg of brinjals.

∴ The total sales of the peasant will be,

$$= ₹ (2000X_1 + 0.75 \times 3000X_2 + 2 \times 1000X_3)$$

∴ Fertilizer expenditure will be,

$$= ₹ 20 (5X_1 + 6X_2 + 5X_3)$$

∴ Peasant's profit will be,

$$Z = \text{Sale (in ₹)} - \text{Total expenditure (in ₹)}$$

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$$= (2000X_1 + 0.75 \times 3000X_2 + 2 \times 1000X_3) - 0.5 \times [100(X_1 + X_2) + 50X_2]$$

$$-20 \times (5X_1 + 6X_2 + 5X_3)$$

$$Z = 1850X_1 + 2080X_2 + 1875X_3$$

Since the total area of the farm is restricted to 100 acres,

$$X_1 + X_2 + X_3 \leq 100$$

Also, the total man-days manpower is restricted to 400 man-days.

$$5X_1 + 6X_2 + 5X_3 \leq 400$$

Hence, the peasant's allocation problem can be finally put in the form:

Find the value of  $X_1$ ,  $X_2$  and  $X_3$  so as to maximize,

$$Z = 1850X_1 + 2080X_2 + 1875X_3$$

Subject to,

$$X_1 + X_2 + X_3 \leq 100$$

$$5X_1 + 6X_2 + 5X_3 \leq 400$$

$$X_1, X_2, X_3 \geq 0$$

**Example 4.8:** ABC Company produces two products: juicers and washing machines. Production happens in two different departments, I and II. Juicers are made in Department I and washing machines in Department II. These two items are sold weekly. The weekly production should not cross 25 juicers and 35 washing machines. The organization always employs a total of 60 employees in the two departments. A juicer requires two man-weeks' labour, while a washing machine needs one man-week's labour. A juicer makes a profit of ₹ 60 and a washing machine contributes a profit of ₹ 40. How many units of juicers and washing machines should the organization make to achieve the maximum profit? Formulate this as an LPP.

**Solution:** Let  $X_1$  and  $X_2$  be the number of units of juicers and washing machines to be produced.

Each juicer and washing machine contributes a profit of ₹ 60 and ₹ 40. Hence, the objective function is to maximize  $Z = 60X_1 + 40X_2$ .

There are two constraints which are imposed: weekly production and labour.

Since the weekly production cannot exceed 25 juicers and 35 washing machines,

$$X_1 \leq 25$$

$$X_2 \leq 35$$

A juicer needs two man-weeks of hard work and a washing machine needs one man-week of hard work and the total number of workers is 60.

$$2X_1 + X_2 \leq 60$$

Non-negativity restrictions: Since the number of juicers and washing machines produced cannot be negative, we have  $X_1 \geq 0$  and  $X_2 \geq 0$ .

Hence, the production of juicers and washing machines problem can be finally put in the form of an LP model given as follows:

Find the value of  $X_1$  and  $X_2$  so as to maximize,

$$Z = 60X_1 + 40X_2$$

Subject to,

$$X_1 \leq 25$$

$$X_2 \leq 35$$

$$2X_1 + X_2 \leq 60$$

$$\text{and } X_1, X_2 \geq 0$$

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### 4.3 CANONICAL AND STANDARD FORMS OF LPP

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The general LPP can be expressed in the following forms, namely canonical or standard forms.

In the standard form, irrespective of the objective function, namely maximize or minimize, all the constraints are expressed as equations. Moreover, RHS of each constraint and all variables are non-negative.

#### Characteristics of the standard form

- (i) The objective function is of maximization type.
- (ii) All constraints are expressed as equations.
- (iii) The right hand side of each constraint is non-negative.
- (iv) All variables are non-negative.

#### Characteristics of the canonical form

- (i) The objective function is of maximization type.
- (ii) All constraints are of ' $\leq$ ' type.
- (iii) All variables  $X_i$  are non-negative.

In the canonical form, if the objective function is of maximization, all the constraints other than non-negativity conditions are ' $\leq$ ' type. If the objective function is of minimization, all the constraints other than non-negative conditions are ' $\geq$ ' type.

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### Notes:

1. Minimization of a function  $Z$  is equivalent to maximization of the negative expression of this function, i.e.,  $\text{Min } Z = -\text{Max } (-Z)$ .
2. An inequality in one direction can be converted into an inequality in the opposite direction by multiplying both sides by  $(-1)$ .
3. Suppose you have the constraint equation,

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

This equation can be replaced by two weak inequalities in opposite directions.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \geq b_1$$

4. If a variable is unrestricted in sign, then it can be expressed as the difference of two non-negative variables, i.e., if  $X_1$  is unrestricted in sign, then  $X_1 = X'_1 - X''_1$ , where  $X_1, X'_1, X''_1$  are  $\geq 0$ .
5. In the standard form, all constraints are expressed as equations, which is possible by introducing some additional variables called slack variables and surplus variables so that a system of simultaneous linear equations is obtained. The necessary transformation will be made to ensure that  $b_i \geq 0$ .

### Definition

- (i) If the constraints of a general LPP be,

$$\sum_{j=1}^n a_{ij} x_j \leq b_i \quad (i = 1, 2, \dots, m),$$

then the non-negative variables  $S_i$ , which are introduced to convert the inequalities ( $\leq$ ) to the equalities  $\sum_{j=1}^n a_{ij} x_j + S_i = b_i \quad (i = 1, 2, \dots, m)$ , are called slack variables.

Slack variables are also defined as the non-negative variables which are added in the LHS of the constraint to convert the inequality ' $\leq$ ' into an equation.

- (ii) If the constraints of a general LPP be,

$$\sum_{j=1}^n a_{ij} x_j \geq b_i \quad (i = 1, 2, \dots, m),$$

then the non-negative variables  $S_i$  which are introduced to convert the inequalities ' $\geq$ ' to the equalities  $\sum_{j=1}^n a_{ij} x_j - S_i = b_i \quad (i = 1, 2, \dots, m)$  are called surplus variables.

Surplus variables are defined as the non-negative variables which are removed from the LHS of the constraint to convert the inequality ' $\geq$ ' into an equation.

### Check Your Progress

1. List the various fields where LP problems are frequently applied.
2. Define 'linearity'.
3. What does the assumption 'finite choices' in an LP problem imply?
4. List the various characteristics of the canonical form.

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## 4.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. LP has been successfully applied in the following fields:
  - (i) Agriculture: LP can be applied in farm management problems as it relates to the allocation of resources such as acreage, labour, water supply or working capital in such a way that it maximizes net revenue.
  - (ii) Contract awards: Evaluation of tenders by recourse to LP guarantees that the awards are made in the cheapest way.
  - (iii) Industries: Applications of LP in business and industry are of the most diverse type. Transportation problems concerning cost minimization can be solved by this technique. The technique can also be adopted in solving problems of production (product-mix) and inventory control.
2. The term linearity implies straight line or proportional relationships among the relevant variables.
3. Finite choices assumption implies that finite number of choices are available to a decision-maker and the decision variables do not assume negative values.
4. The following are the various characteristics of the canonical form:
  - (i) The objective function is of maximization type.
  - (ii) All constraints are of ' $\leq$ ' type.
  - (iii) All variables  $X_i$  are non-negative.

## 4.5 SUMMARY

- Linear programming is a decision-making technique under given constraints on the assumption that the relationships among the variables representing different phenomena happen to be linear.
- LP is the most widely used technique of decision-making in business and industry. Thus, it finds application in production planning and scheduling, transportation, sales and advertising, financial planning, portfolio analysis, corporate planning and many other fields.

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- Criterion function, which is also known as objective function, is a component of LP problems. It states whether the determinants of the quantity should be maximized or minimized.
- Feasible solutions are all those possible solutions which can be worked upon under given constraints.
- Simple linear LP problems with two decision variables can be easily solved using the graphical method.
- In the standard form of LPP, all constraints are expressed as equations; while in the standard form all constraints are expressed as equations by introducing additional variables called slack variables and surplus variables.

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### 4.6 KEY WORDS

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- **Constraints:** Restrictions or limitations imposed on decision variables
- **Decision variables:** Variables that form the objective function and on which the cost or profit depends

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### 4.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

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#### Short Answer Questions

1. What is proportionality in linear programming?
2. What do you understand by certainty in linear programming?
3. What are the basic constituents of an LP model?

#### Long Answer Questions

1. A company manufactures 3 products *A*, *B* and *C*. The profits are: ₹ 3, ₹ 2 and ₹ 4 respectively. The company has two machines and given below is the required processing time in minutes for each machine on each product.

	<i>Products</i>		
<i>Machines</i>	<i>A</i>	<i>B</i>	<i>C</i>
I	4	3	5
II	2	2	4

Machines I and II have 2000 and 2500 minutes, respectively. The company must manufacturer, 100 *A*'s, 200 *B*'s and 50 *C*'s but no more than 150 *A*'s. Find the number of units of each product to be manufactured by the company to maximize the profit. Formulate the above as an LP Model.

2. A company produces two types of leather belts *A* and *B*. *A* is of superior quality and *B* is of inferior quality. The respective profits are ₹ 10 and ₹ 5

per belt. The supply of raw materials is sufficient for making 850 belts per day. For belt  $A$ , a special type of buckle is required and 500 are available per day. There are 700 buckles available for belt  $B$  per day. Belt  $A$  needs twice as much time as that required for belt  $B$  and the company can produce 500 belts if all of them were of the type  $A$ . Formulate an LP Model for the above problem.

3. The standard weight of a special purpose brick is 5 kg and it contains two ingredients  $B_1$  and  $B_2$ , where  $B_1$  costs ₹ 5 per kg and  $B_2$  costs ₹ 8 per kg. Strength considerations dictate that the brick contains not more than 4 kg of  $B_1$  and a minimum of 2 kg of  $B_2$  since the demand for the product is likely to be related to the price of the brick. Formulate the above problem as an LP model.

## NOTES

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### 4.8 FURTHER READINGS

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## BLOCK - II

### LPP

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#### NOTES

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## UNIT 5 SIMPLE REGRESSION AND CORRELATION ANALYSIS

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### Structure

- 5.0 Introduction
- 5.1 Objectives
- 5.2 Introduction to Correlation Analysis
  - 5.2.1 Coefficient of Correlation and Linear Regression
- 5.3 Solving LP Problems: Graphical and Simplex Method
  - 5.3.1 Procedure for Solving LPP by Graphical Method
  - 5.3.2 Simplex Method
- 5.4 Concept of Duality and Sensitivity Analysis
  - 5.4.1 Formulation of a Dual Problem; 5.4.2 Definition of a Dual Problem
  - 5.4.3 Sensitivity Analysis; 5.4.4 Shadow Price
  - 5.4.5 Economic Interpretation: Interpreting the Solution for Decision-Making
- 5.5 Answers to Check Your Progress Questions
- 5.6 Summary
- 5.7 Key Words
- 5.8 Self Assessment Questions and Exercises
- 5.9 Further Readings

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### 5.0 INTRODUCTION

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In this unit, you will learn about the correlation analysis techniques that analyses the indirect relationships in sample survey data and establishes the variables which are most closely associated with a given action or mindset.

You will also learn about the standard and canonical forms of LPP. There are certain special cases in linear programming, about which you will learn, including degeneracy, non-feasible solutions and alternate optima. You will learn about duality and sensitivity analysis and learn to formulate a dual problem.

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### 5.1 OBJECTIVES

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After going through this unit, you will be able to:

- Understand what is correlation
- Explain coefficient of correlation
- Solve an LP problems using the simplex method
- Understand the concept of duality and sensitivity analysis



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## 5.2 INTRODUCTION TO CORRELATION ANALYSIS

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Correlation analysis is the statistical tool generally used to describe the degree to which one variable is related to another. The relationship, if any, is usually assumed to be a linear one. This analysis is used quite frequently in conjunction with regression analysis to measure how well the regression line explains the variations of the dependent variable. In fact, the word correlation refers to the relationship or interdependence between two variables. There are various phenomena which have relation to each other. When, for instance, demand of a certain commodity increases, then its price goes up and when its demand decreases then its price comes down. Similarly, with age the height of the children, with height the weight of the children, with money supply the general level of prices go up. Such sort of relationship can as well be noticed for several other phenomena. The theory by means of which quantitative connections between two sets of phenomena are determined is called the *Theory of Correlation*.

On the basis of the theory of correlation one can study the comparative changes occurring in two related phenomena and their cause-effect relation can be examined. It should, however, be borne in mind that relationship like ‘black cat causes bad luck’, ‘filled-up pitchers result in good fortune’ and similar other beliefs of the people cannot be explained by the theory of correlation since they are all imaginary and are incapable of being justified mathematically. Thus, correlation is concerned with the relationship between two related and quantifiable variables. If two quantities vary in sympathy so that a movement (an increase or decrease) in the one tends to be accompanied by a movement in the same or opposite direction in the other and the greater the change in the one, the greater is the change in the other, the quantities are said to be correlated. This type of relationship is known as correlation or what is sometimes called, in statistics, as co-variation.

For correlation it is essential that the two phenomena should have a cause-effect relationship. If such relationship does not exist then there can be no correlation. If, for example, the height of the students as well as the height of the trees increases, then one should not call it a case of correlation because the two phenomena, viz. the height of students and the height of trees are not even causally related. However, the relationship between the price of a commodity and its demand, the price of a commodity and its supply, the rate of interest and savings, etc. are examples of correlation since in all such cases the change in one phenomenon is explained by a change in other phenomenon.

It is appropriate here to mention that correlation in case of phenomena pertaining to natural sciences can be reduced to absolute mathematical terms, e.g., heat always increases with light. But in phenomena pertaining to social sciences it is often difficult to establish any absolute relationship between two phenomena. Hence, in social sciences we must take the fact of correlation being established if

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in a large number of cases, two variables always tend to move in the same or opposite direction.

*Correlation can either be positive or it can be negative.* Whether correlation is positive or negative would depend upon the direction in which the variables are moving. If both variables are changing in the same direction, then correlation is said to be positive but when the variations in the two variables take place in opposite direction, the correlation is termed as negative. This can be explained as follows:

<i>Changes in Independent Variable</i>	<i>Changes in Dependent Variable</i>	<i>Nature of Correlation</i>
Increase (+)↑	Increase (+)↑	Positive (+)
Decrease (-)↓	Decrease (-)↓	Positive (+)
Increase (+)↑	Decrease (-)↓	Negative (-)
Decrease (-)↓	Increase (+)↑	Negative (-)

*Correlation can either be linear or it can be non-linear.* The non-linear correlation is also known as curvilinear correlation. The distinction is based upon the constancy of the ratio of change between the variables. When the amount of change in one variable tends to bear a constant ratio to the amount of change in the other variable, then the correlation is said to be linear. In such a case if the values of the variables are plotted on a graph paper, then a straight line is obtained. This is why the correlation is known as linear correlation. But when the amount of change in one variable does not bear a constant ratio to the amount of change in the other variable, i.e., the ratio happens to be a variable instead of a constant, then the correlation is said to be non-linear or curvilinear. In such a situation we shall obtain a curve if the values of the variables are plotted on a graph paper.

*Correlation can either be simple correlation or it can be partial correlation or multiple correlation.* The study of correlation for two variables (of which one is independent and the other is dependent) involves application of simple correlation. When more than two variables are involved in a study relating to correlation, then it can either be a multiple correlation or a partial correlation. Multiple correlation studies the relationship between a dependent variable and two or more independent variables. In partial correlation, we measure the correlation between a dependent variable and one particular independent variable assuming that all other independent variables remain constant.

Statisticians have developed two measures for describing the correlation between two variables, viz. the coefficient of determination and the coefficient of correlation.

**5.2.1 Coefficient of Correlation and Linear Regression**

The coefficient of correlation symbolically denoted by ‘ $r$ ’ is another important measure to describe how well one variable is explained by another. It measures the degree of relationship between the two causally-related variables. The value of this coefficient can never be more than +1 or less than -1. Thus +1 and -1 are the limits of this coefficient. For a unit change in independent variable, if there

happens to be a constant change in the dependent variable in the same direction, then the value of the coefficient will be +1 indicative of the perfect positive correlation; but if such a change occurs in the opposite direction, the value of the coefficient will be -1, indicating the perfect negative correlation. In practical life the possibility of obtaining either a perfect positive or perfect negative correlation is very remote particularly in respect of phenomena concerning social sciences. If the coefficient of correlation has a zero value then it means that there exists no correlation between the variables under study.

There are several methods of finding the coefficient of correlation but the following ones are considered important:

- (i) Coefficient of Correlation by the Method of Least Squares
- (ii) Coefficient of Correlation using Simple Regression Coefficients
- (iii) Coefficient of Correlation through Product Moment Method or Karl Pearson's Coefficient of Correlation

Whichever of these above-mentioned three methods we adopt, we get the same value of  $r$ .

### Coefficient of Correlation by the Method of Least Squares

Under this method, first of all the estimating equation is obtained using the least square method of simple regression analysis. The equation is worked out as:

$$\hat{Y} = a + bX_i$$

Total variation	$= \Sigma (Y - \bar{Y})^2$
Unexplained variation	$= \Sigma (Y - \hat{Y})^2$
Explained variation	$= \Sigma (\hat{Y} - \bar{Y})^2$

Then, by applying the following formulae we can find the value of the coefficient of correlation.

$$\begin{aligned} r &= \sqrt{r^2} = \sqrt{\frac{\text{Explained variation}}{\text{Total variation}}} \\ &= \sqrt{1 - \frac{\text{Unexplained variation}}{\text{Total variation}}} \\ &= \sqrt{1 - \frac{\Sigma (Y - \hat{Y})^2}{\Sigma (Y - \bar{Y})^2}} \end{aligned}$$

*This clearly shows that coefficient of correlation happens to be the squareroot of the coefficient of determination.*

Short-cut formula for finding the value of 'r' by the method of least squares can be repeated and readily written as follows:

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Where

$$r = \sqrt{\frac{a\Sigma Y + b\Sigma XY - n\bar{Y}^2}{\Sigma Y^2 - n\bar{Y}^2}}$$

$a$  =  $Y$ -intercept

$b$  = Slope of the estimating equation

$X$  = Values of the independent variable

$Y$  = Values of dependent variable

$\bar{Y}$  = Mean of the observed values of  $Y$

$n$  = Number of items in the sample

(i.e., pairs of observed data)

The plus (+) or the minus (–) sign of the coefficient of correlation worked out by the method of least squares is related to the sign of ‘ $b$ ’ in the estimating equation, viz.  $\hat{Y} = a + bX_i$ . If ‘ $b$ ’ has a minus sign, the sign of ‘ $r$ ’ will also be minus but if ‘ $b$ ’ has a plus sign, then the sign of ‘ $r$ ’ will also be plus. The value of ‘ $r$ ’ indicates the degree along with the direction of the relationship between the two variables  $X$  and  $Y$ .

### Coefficient of Correlation using Simple Regression Coefficient

Under this method, the estimating equation of  $Y$  and the estimating equation of  $X$  is worked out using the method of least squares. From these estimating equations we find the regression coefficient of  $X$  on  $Y$ , i.e., the slope of the estimating equation of  $X$  (symbolically written as  $b_{XY}$ ) and this is equal to  $r \frac{\sigma_X}{\sigma_Y}$  and similarly, we find the regression coefficient of  $Y$  on  $X$ , i.e. the slope of the estimating equation of  $Y$  (symbolically written as  $b_{YX}$ ) and this is equal to  $r \frac{\sigma_Y}{\sigma_X}$ . For finding ‘ $r$ ’, the square root of the product of these two regression coefficients are worked out as stated below:<sup>1</sup>

$$\begin{aligned} r &= \sqrt{b_{XY} \cdot b_{YX}} &= \sqrt{r \frac{\sigma_X}{\sigma_Y} \cdot r \frac{\sigma_Y}{\sigma_X}} \\ &= \sqrt{r^2} &= r \end{aligned}$$

As stated earlier, the sign of ‘ $r$ ’ will depend upon the sign of the regression coefficients. If they have minus sign, then ‘ $r$ ’ will take minus sign but the sign of ‘ $r$ ’ will be plus if regression coefficients have plus sign.

### Karl Pearson’s Coefficient of Correlation

Karl Pearson’s method is the most widely-used method of measuring the relationship between two variables. This coefficient is based on the following assumptions:

- There is a linear relationship between the two variables which means that straight line would be obtained if the observed data are plotted on a graph.
- The two variables are causally related which means that one of the variables is independent and the other one is dependent.
- A large number of independent causes are operating in both the variables so as to produce a normal distribution.

According to Karl Pearson, 'r' can be worked out as under:

$$r = \frac{\sum XY}{n\sigma_X\sigma_Y}$$

where

$$X = (X - \bar{X})$$

$$Y = (Y - \bar{Y})$$

$\sigma_X$  = Standard deviation of

X series and is equal to  $\sqrt{\frac{\sum X^2}{n}}$

$\sigma_Y$  = Standard deviation of

Y series and is equal to  $\sqrt{\frac{\sum Y^2}{n}}$

n = Number of pairs of X and Y observed.

A short-cut formula known as the Product Moment Formula can be derived from the above-stated formula as under:

$$\begin{aligned} r &= \frac{\sum XY}{n\sigma_X\sigma_Y} = \frac{\sum XY}{\sqrt{\frac{\sum X^2}{n} \cdot \frac{\sum Y^2}{n}}} \\ &= \frac{\sum XY}{\sqrt{\sum X^2 \sum Y^2}} \end{aligned}$$

The above formulae are based on obtaining true means (viz.  $\bar{X}$  and  $\bar{Y}$ ) first and then doing all other calculations. This happens to be a tedious task particularly if the true means are in fractions. To avoid difficult calculations, we make use of the assumed means in taking out deviations and doing the related calculations. In such a situation, we can use the following formula for finding the value of 'r':

**(a) In Case of Ungrouped Data:**

$$\begin{aligned} r &= \frac{\frac{\sum dX \cdot dY}{n} - \left( \frac{\sum dX}{n} \cdot \frac{\sum dY}{n} \right)}{\sqrt{\frac{\sum dX^2}{n} \left( \frac{\sum dX}{n} \right)^2} \cdot \sqrt{\frac{\sum dY^2}{n} \left( \frac{\sum dY}{n} \right)^2}} \\ &= \frac{\sum dX \cdot dY - \left( \frac{\sum dX \times \sum dY}{n} \right)}{\sqrt{\sum dX^2 - \frac{(\sum dX)^2}{n}} \sqrt{\sum dY^2 - \frac{(\sum dY)^2}{n}}} \end{aligned}$$

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where  $\Sigma dX = \Sigma(X - X_A)$        $X_A =$  Assumed average of  $X$   
 $\Sigma dY = \Sigma(Y - Y_A)$        $Y_A =$  Assumed average of  $Y$   
 $\Sigma dX^2 = \Sigma(X - X_A)^2$   
 $\Sigma dY^2 = \Sigma(Y - Y_A)^2$   
 $\Sigma dX \cdot dY = \Sigma(X - X_A)(Y - Y_A)$   
 $n =$  Number of pairs of observations of  $X$  and  $Y$

**(b) In Case of Grouped Data**

$$r = \frac{\frac{\Sigma fdX \cdot dY}{n} - \left( \frac{\Sigma fdX}{n} \cdot \frac{\Sigma fdY}{n} \right)}{\sqrt{\frac{\Sigma fdX^2}{n} - \left( \frac{\Sigma fdX}{n} \right)^2} \sqrt{\frac{\Sigma fdY^2}{n} - \left( \frac{\Sigma fdY}{n} \right)^2}}$$

or

$$r = \frac{\Sigma fdX \cdot dY - \left( \frac{\Sigma fdX \cdot \Sigma fdY}{n} \right)}{\sqrt{\Sigma fdX^2 - \left( \frac{\Sigma fdX}{n} \right)^2} \sqrt{\Sigma fdY^2 - \left( \frac{\Sigma fdY}{n} \right)^2}}$$

where  $\Sigma fdX \cdot dY = \Sigma f(X - X_A)(Y - Y_A)$   
 $\Sigma fdX = \Sigma f(X - X_A)$   
 $\Sigma fdY = \Sigma f(Y - Y_A)$   
 $\Sigma fdY^2 = \Sigma f(Y - Y_A)^2$        $\Sigma fdX^2 = \Sigma f(X - X_A)^2$   
 $n =$  Number of pairs of observations of  $X$  and  $Y$

**Probable Error of the Coefficient of Correlation**

Probable Error (PE) of  $r$  is very useful in interpreting the value of  $r$  and is worked out as under for Karl Pearson's coefficient of correlation:

$$PE = 0.6745 \frac{1 - r^2}{\sqrt{n}}$$

If  $r$  is less than its PE, it is not at all significant. If  $r$  is more than PE, there is correlation. *If  $r$  is more than 6 times its PE and greater than  $\pm 0.5$ , then it is considered significant.*

**Example 5.1:** From the following data calculate ' $r$ ' between  $X$  and  $Y$  applying the following three methods:

- (a) The method of least squares
- (b) The method based on regression coefficients
- (c) The product moment method of Karl Pearson

Verify the obtained result of any one method with that of another.

$X$	1	2	3	4	5	6	7	8	9
$Y$	9	8	10	12	11	13	14	16	15

**Solution:** Let us develop the following table for calculating the value of ‘ $r$ ’:

$X$	$Y$	$X^2$	$Y^2$	$XY$
1	9	1	81	9
2	8	4	64	16
3	10	9	100	30
4	12	16	144	48
5	11	25	121	55
6	13	36	169	78
7	14	49	196	98
8	16	64	256	128
9	15	81	225	135

$n=9$

$\Sigma X=45$	$\Sigma Y=108$	$\Sigma X^2=285$	$\Sigma Y^2=1356$	$\Sigma XY=597$
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$$\therefore \bar{X}=5; \quad \bar{Y}=12$$

**(i) Coefficient of Correlation by the Method of Least Squares is Worked out as Under:**

First of all find out the estimating equation

$$\hat{Y} = a + bX_i$$

where

$$b = \frac{\Sigma XY - n\bar{X}\bar{Y}}{\Sigma X^2 - n\bar{X}^2}$$

$$= \frac{597 - 9(5)(12)}{285 - 9(25)} = \frac{597 - 540}{285 - 225} = \frac{57}{60} = 0.95$$

and

$$a = \bar{Y} - b\bar{X}$$

$$= 12 - 0.95(5) = 12 - 4.75 = 7.25$$

Hence,

$$\hat{Y} = 7.25 + 0.95X_i$$

Now, ‘ $r$ ’ can be worked out as under by the method of least squares:

$$r = \sqrt{1 - \frac{\text{Unexplained variation}}{\text{Total variation}}}$$

$$= \sqrt{1 - \frac{\Sigma(Y - \hat{Y})^2}{\Sigma(Y - \bar{Y})^2}} = \sqrt{\frac{\Sigma(\hat{Y} - \bar{Y})^2}{\Sigma(Y - \bar{Y})^2}}$$

$$= \sqrt{\frac{a\Sigma Y + b\Sigma XY - n\bar{Y}^2}{\Sigma Y^2 - n\bar{Y}^2}}$$

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As per the short-cut formula,

$$\begin{aligned} r &= \sqrt{\frac{7.25(108) + 0.95(597) - 9(12)^2}{1356 - 9(12)^2}} \\ &= \sqrt{\frac{783 + 567.15 - 1296}{1356 - 1296}} = \sqrt{\frac{54.15}{60}} \\ &= \sqrt{0.9025} = 0.95 \end{aligned}$$

**(ii) Coefficient of Correlation by the Method Based on Regression Coefficients is Worked out as Under:**

Regression coefficients of  $Y$  on  $X$ :

$$\begin{aligned} b_{YX} &= \frac{\sum XY - n\bar{X}\bar{Y}}{\sum X^2 - n\bar{X}^2} \\ &= \frac{597 - 9 \times 5 \times 12}{285 - 9(5)^2} = \frac{597 - 540}{285 - 225} = \frac{57}{60} \end{aligned}$$

Regression coefficient of  $X$  on  $Y$ :

$$\begin{aligned} b_{XY} &= \frac{\sum XY - n\bar{X}\bar{Y}}{\sum Y^2 - n\bar{Y}^2} \\ &= \frac{597 - 9 \times 5 \times 12}{1356 - 9(12)^2} = \frac{597 - 540}{1356 - 1296} = \frac{57}{60} \end{aligned}$$

Hence,

$$r = \sqrt{b_{YX} \cdot b_{XY}} = \sqrt{\frac{57}{60} \times \frac{57}{60}} = \frac{57}{60} = 0.95$$

**(iii) Coefficient of Correlation by the Product Moment Method of Karl Pearson is Worked out as Under:**

$$\begin{aligned} r &= \frac{\sum XY - n\bar{X}\bar{Y}}{\sqrt{\sum X^2 - n\bar{X}^2} \sqrt{\sum Y^2 - n\bar{Y}^2}} \\ &= \frac{597 - 9(5)(12)}{\sqrt{285 - 9(5)^2} \sqrt{1356 - 9(12)^2}} \\ &= \frac{597 - 540}{\sqrt{285 - 225} \sqrt{1356 - 1296}} = \frac{57}{\sqrt{60} \sqrt{60}} = \frac{57}{60} = 0.95 \end{aligned}$$

Hence, we get the value of  $r = 0.95$ . We get the same value applying the other two methods also. Therefore, whichever method we apply, the results will be the same.



**Example 5.2:** Calculate the coefficient of correlation and lines of regression from the following data.

Y Sales Revenue (Rs '000)	X Advertising Expenditure (Rs '00)				Total
	5-15	15-25	25-35	35-45	
75-125	3	4	4	8	19
125-175	8	6	5	7	26
175-225	2	2	3	4	11
225-275	2	3	2	2	9
Total	15	15	14	21	n = 65

**NOTES**

**Solution:** Since the given information is a case of bivariate grouped data we shall extend the given table rightwards and downwards to obtain various values for finding 'r' as stated below:

Y Sales Revenue (Rs '000)	X Advertising Expenditure (Rs '00)				Total	Midpoint of Y If A = 200 i = 50 ∴ dY	fdY	fdY <sup>2</sup>	fdX	fdX <sup>2</sup>	fdX.dY
	5-15	15-25	25-35	35-40							
75-125	3	4	4	8	19	100	-2	-38	76	4	
125-175	8	6	5	7	26	150	-1	-26	26	15	
175-225	2	2	3	4	11	200	0	0	0	0	
225-275	2	3	2	2	9	250	1	9	9	-5	
Total	15	15	14	21	n = 65						ΣfdY ΣfdY <sup>2</sup>
ΣfdX.dY (or f)											= -55 = 111 = 14
Midpoint of X If A = 30 i = 10 ∴ dX	10	20	30	40							
fdX	-30	-15	0	21	ΣfdX						= -24
fdX <sup>2</sup>	60	15	0	21	ΣfdX <sup>2</sup>						= 96
fdX.dY	24*	11	0	-21	ΣfdX.dY						= 14

$$\therefore r = \frac{\frac{\sum fdX.dY}{n} - \left(\frac{\sum fdX}{n} \frac{\sum fdY}{n}\right)}{\sqrt{\frac{\sum fdX^2}{n} - \left(\frac{\sum fdX}{n}\right)^2} \sqrt{\frac{\sum fdY^2}{n} - \left(\frac{\sum fdY}{n}\right)^2}}$$

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Putting the calculated values in the above equation we have:

$$r = \frac{\frac{14}{65} - \left( \frac{-24}{65} \times \frac{-55}{65} \right)}{\sqrt{\frac{96}{65} - \left( \frac{-24}{65} \right)^2} \sqrt{\frac{111}{65} - \left( \frac{-55}{65} \right)^2}}$$

$$= \frac{0.2154 - (+0.3124)}{\sqrt{1.48 - 0.14} \sqrt{1.71 - 0.72}}$$

$$= \frac{(-)0.0970}{\sqrt{(1.34) \times (99)}} = \frac{-0.00970}{\sqrt{1.3266}} = \frac{-0.0970}{1.15} = (-)0.0843$$

Hence,

$$r = (-)0.0843$$

This Shows a poor negative correlation between the two variables. Since only 0.64% ( $r^2 (0.08)^2 = 0.0064$ ) variation in  $Y$  (Sales revenue) is explained by variation in  $X$  (Advertising expenditure).

The two lines of regression are as under:

Regression line of  $X$  on  $Y$ :  $(X - \bar{X}) = r \frac{\sigma_X}{\sigma_Y} (Y - \bar{Y})$

Regression line of  $Y$  on  $X$ :  $(Y - \bar{Y}) = r \frac{\sigma_Y}{\sigma_X} (X - \bar{X})$

First obtain the following values:

$$\bar{X} = A + \frac{\sum fdX}{n} . i = 30 + \frac{(-24)}{65} \times 10 = 26.30$$

$$\bar{Y} = A + \frac{\sum fdY}{n} . i = 200 + \frac{-55}{65} \times 10 = 157.70$$

$$\sigma_X = \sqrt{\frac{\sum fdX^2}{n} - \left( \frac{\sum fdX}{n} \right)^2} \times i = \sqrt{\frac{96}{65} - \left( \frac{-24}{65} \right)^2} \times 10 = 11.60$$

$$\sigma_Y = \sqrt{\frac{\sum fdY^2}{n} - \left( \frac{\sum fdY}{n} \right)^2} \times i = \sqrt{\frac{111}{65} - \left( \frac{-55}{65} \right)^2} \times 50 = 49.50$$

Therefore, the regression line of  $X$  on  $Y$ :

$$(X - 26.30) = \frac{11.6}{49.5} (-0.084) (Y - 157.70)$$

or

$$\hat{X} = -0.02Y + 3.15 + 26.30$$

$\therefore$

$$\hat{X} = -0.02Y + 29.45$$

Regression line of  $Y$  on  $X$ :

$$(Y - 157.70) = \frac{49.5}{11.6} (-0.084) (X - 26.30)$$

or,

$$\hat{Y} = -0.36X + 9.47 + 157.70$$

$\therefore$

$$\hat{Y} = -0.36X + 167.17$$

### 5.3 SOLVING LP PROBLEMS: GRAPHICAL AND SIMPLEX METHOD

Simple linear programming problems with two decision variables can be easily solved by graphical method.

#### 5.3.1 Procedure for Solving LPP by Graphical Method

The steps involved in the graphical method are as follows.

**Step 1:** Consider each inequality constraint as an equation.

**Step 2:** Plot each equation on the graph as each will geometrically represent a straight line.

**Step 3:** Mark the region. If the inequality constraint corresponding to that line is  $\leq$ , then the region below the line lying in the first quadrant (due to non-negativity of variables) is shaded. For the inequality constraint  $\geq$  sign, the region above the line in the first quadrant is shaded. The points lying in the common region will satisfy all the constraints simultaneously. The common region, thus obtained, is called the feasible region.

**Step 4:** Allocate an arbitrary value, say zero, for the objective function.

**Step 5:** Draw a straight line to represent the objective function with the arbitrary value (i.e., a straight line through the origin).

**Step 6:** Stretch the objective function line till the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin and passes through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passes through at least one corner of the feasible region.

**Step 7:** Find the coordinates of the extreme points selected in Step 6 and find the maximum or minimum value of  $Z$ .

**Note:** As the optimal values occur at the corner points of the feasible region, it is enough to calculate the value of the objective function of the corner points of the feasible region and select the one which gives the optimal solution, i.e., in the case of a maximization problem, the optimal point corresponds to the corner point at which the objective function has the maximum value and in the case of minimization, the corner point which gives the objective function the minimum value is the optimal solution.

**Example 5.3:** Solve the following LPP by the graphical method.

$$\text{Minimize } Z = 20X_1 + 10X_2$$

$$\text{Subject to, } X_1 + 2X_2 \leq 40$$

$$3X_1 + X_2 \geq 30$$

$$4X_1 + 3X_2 \geq 60$$

$$X_1, X_2 \geq 0$$

### NOTES

**NOTES**

**Solution:** Replace all the inequalities of the constraints by equation,

$$X_1 + 2X_2 = 40 \quad \text{If } X_1 = 0 \Rightarrow X_2 = 20$$

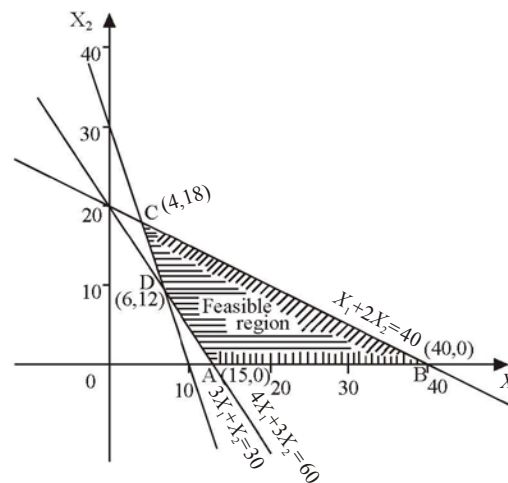
$$\text{If } X_2 = 0 \Rightarrow X_1 = 40$$

$$\therefore X_1 + 2X_2 = 40 \text{ passes through } (0, 20) (40, 0).$$

$$3X_1 + X_2 = 30 \text{ passes through } (0, 30) (10, 0).$$

$$4X_1 + 3X_2 = 60 \text{ passes through } (0, 20) (15, 0).$$

Plot each equation on the graph.



The feasible region is  $ABCD$ .

$C$  and  $D$  are points of intersection of lines.

$$X_1 + 2X_2 = 40, \quad 3X_1 + X_2 = 30$$

$$\text{And, } 4X_1 + 3X_2 = 60$$

On solving, we get  $C = (4, 18)$  and  $D = (6, 12)$

*Corner Points*      *Value of  $Z = 20X_1 + 10X_2$*

$$A (15, 0) \quad 300$$

$$B (40, 0) \quad 800$$

$$C (4, 18) \quad 260$$

$$D (6, 12) \quad 240 \text{ (Minimum value)}$$

$\therefore$  The minimum value of  $Z$  occurs at  $D (6, 12)$ . Hence, the optimal solution is  $X_1 = 6, X_2 = 12$ .

**5.3.2 Simplex Method**

The **simplex method** is an iterative procedure for solving an LPP in a finite number of steps. This method provides an algorithm which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more as the case may be at the previous vertex. This procedure is repeated and since the number of

vertices is finite, the method leads to an optimal vertex in a finite number of steps or indicates the existence of an unbounded solution.

### Definition

- (i) Let  $X_B$  be a basic feasible solution to the LPP.

$$\text{Max } Z = C_X$$

Subject to  $A_X = b$  and  $X \geq 0$ , such that it satisfies  $X_B = B^{-1}b$ ,

Here  $B$  is the basic matrix formed by the column of basic variables.

The vector  $C_B = (C_{B1}, C_{B2}, \dots, C_{Bm})$ , where  $C_{Bj}$  are components of  $C$  associated with the basic variables is called the cost vector associated with the basic feasible solution  $X_B$ .

- (ii) Let  $X_B$  be a basic feasible solution to the LPP.

$$\text{Max } Z = C_X, \text{ where } A_X = b \text{ and } X \geq 0.$$

Let  $C_B$  be the cost vector corresponding to  $X_B$ . For each column vector  $a_j$  in  $A_1$ , which is not a column vector of  $B$ , let

$$a_j = \sum_{i=1}^m a_{ij} b_i$$

Then the number  $Z_j = \sum_{i=1}^m C_{Bi} a_{ij}$  is called the evaluation corresponding to

$a_j$  and the number  $(Z_j - C_j)$  is called the net evaluation corresponding to  $j$ .

### Simplex Algorithm

While solving an LPP by simplex algorithm, an important assumption is the existence of an initial basic feasible solution. The following are the steps involved for finding an optimum solution.

**Step 1:** It is first imperative to find out whether minimization or maximization is the objective function of the given LPP. If minimization is the objective function, then the LPP should be converted into a maximization problem,

$$\text{Min } Z = -\text{Max } (-Z)$$

**Step 2:** All values of  $b_i$  ( $i = 1, 2, \dots, m$ ) should be positive. Any negative  $b_i$  value present should be converted into positive value by multiplying with  $-1$ .

**Step 3:** The problem should be then expressed in the standard form. This can be done by introducing slack/surplus variables to convert the inequality constraints into equations.

**Step 4:** Get an initial basic feasible solution to the problem in the form  $X_B = B^{-1}b$  and put it in the first column of the simplex table. Form the initial simplex table shown as follows:

### NOTES

**NOTES**

		$C_j$	$C_1$	$C_2$	$C_3$	$C_4$	..... $C_n$	0 0 ..... 0
$C_B$	$S_B$	$X_B$	$X_1$	$X_2$	$X_3$	$X_4$	..... $X_n$	$S_1 S_2 \dots S_m$
$C_{B1}$	$S_1$	$b_1$	$a_{11}$	$a_{12}$	$a_{13}$	$a_{14}$	..... $a_{1n}$	1 0 ..... 0
$C_{B2}$	$S_2$	$b_2$	$a_{21}$	$a_{22}$	$a_{23}$	$a_{24}$	..... $a_{2n}$	1 0 ..... 0

**Step 5:** Compute the net evaluations  $Z_j - C_j$  by using the relation:

$$Z_j - C_j = C_B (a_j - C_j)$$

Examine the sign of  $Z_j - C_j$ :

- (i) If all  $Z_j - C_j \geq 0$ , then the initial basic feasible solution  $X_B$  is an optimum basic feasible solution.
- (ii) If at least one  $Z_j - C_j > 0$ , then go to the next step as the solution is not optimal.

**Step 6:** To find the entering variable, i.e., key column.

If there are more than one negative  $Z_j - C_j$ , the most negative of them should be chosen. Let it be  $Z_r - C_r$  for some  $j=r$ . This gives the entering variable  $X_r$  and is indicated by an arrow at the bottom of the  $r$ th column. If there are more than one variables having the same most negative  $Z_j - C_j$ , then any one of the variable can be selected arbitrarily as the entering variable.

- (i) If all  $X_{ir} \leq 0$  ( $i = 1, 2, \dots, m$ ), then there is an unbounded solution to the given problem.
- (ii) If at least one  $X_{ir} > 0$  ( $i = 1, 2, \dots, m$ ), then the corresponding vector  $X_r$  enters the basis.

**Step 7:** To find the leaving variable or key row:

Compute the ratio ( $X_{Bi}/X_{kr}, X_{ir} > 0$ )

If the minimum of these ratios be  $X_{Bi}/X_{kr}$ , then choose the variable  $X_k$  to leave the basis called the key row and the element at the intersection of the key row and the key column is called the key element.

**Step 8:** Form a new basis by dropping the leaving variable and introducing the entering variable along with the associated value under  $C_B$  column. The leaving element is converted to unity by dividing the key equation by the key element and all other elements in its column to zero by using the formula:

New element = Old element

$$-\left[ \frac{\text{Product of elements in the key row and key column}}{\text{Key element}} \right]$$

**Step 9:** Repeat the procedure of Step (5) until either an optimum solution is obtained or there is an indication of unbounded solution.

**Example 5.4:** Use the simplex method to solve the following LPP.

$$\text{Maximize } Z = 3X_1 + 2X_2$$

$$\text{Subject to, } X_1 + X_2 \leq 4$$

$$X_1 - X_2 \leq 2$$

$$X_1, X_2 \geq 0$$

**Solution:** By introducing the slack variables  $S_1, S_2$ , convert the problem into its standard form.

$$\text{Max } Z = 3X_1 + 2X_2 + 0S_1 + 0S_2$$

$$\text{Subject to, } X_1 + X_2 + S_1 = 4$$

$$X_1 - X_2 + S_2 = 2$$

$$X_1, X_2, S_1, S_2 \geq 0$$

$$\begin{bmatrix} X_1 & X_2 & S_1 & S_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

An initial basic feasible solution is given by,

$$X_B = B^{-1}b,$$

$$\text{Here, } B = I_2, X_B = (S_1, S_2)$$

$$\text{i.e., } (S_1, S_2) = I_2 (4, 2) = (4, 2)$$

**Initial Simplex Table**

$$Z_j = C_B a_j$$

$$Z_1 - C_1 = C_B a_1 - C_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 3 = -3$$

$$Z_2 - C_2 = C_B a_2 - C_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (1-1) - 2 = -2$$

$$Z_3 - C_3 = C_B a_3 - C_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (1 \cdot 0) - 0 = -0$$

$$Z_4 - C_4 = C_B a_4 - C_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (0 \cdot 1) - 0 = -0$$

	$C_j$		3	2	0	0	
$C_B$	$B$	$X_B$	$X_1$	$X_2$	$S_1$	$S_2$	Min $\frac{X_B}{X_1}$
0	$S_1$	4	1	1	1	0	4/1 = 4
←0	$S_2$	2	①	-1	0	1	2/1 = 2
	$Z_j$	0	0	0	0	0	
	$Z_j - C_j$		-3↑	-2	0	0	

**NOTES**

**NOTES**

Since there are some  $Z_j - C_j = 0$ , the current basic feasible solution is not optimum.

Since  $Z_1 - C_1 = -3$  is the most negative, the corresponding non-basic variable  $X_1$  enters the basis.

The column corresponding to this  $X_1$  is called the key column.

$$\text{Ratio} = \text{Min} \left\{ \frac{X_{Bi}}{X_{ir}}, X_{ir} > 0 \right\}$$

$$= \text{Min} \left\{ \frac{4}{1}, \frac{2}{1} \right\}, \text{ which corresponds to } S_2$$

$\therefore$  The leaving variable is the basic variable  $S_2$ . This row is called the key row. Convert the leading element  $X_{21}$  to unit and all other elements in its column  $n$ , i.e.,  $(X_1)$  to zero by using the formula:

New element = Old element -

$$\left[ \frac{\text{Product of elements in the key row and key column}}{\text{Key element}} \right]$$

To apply this formula, first we find the ratio, namely

$$\frac{\text{The element to be zero}}{\text{Key element}} = \frac{1}{1} = 1$$

Apply this ratio for the number of elements that are converted in the key row. Multiply this ratio by the key row element shown as follows:

$$1 \times 2$$

$$1 \times 1$$

$$1 \times -1$$

$$1 \times 0$$

$$1 \times 1$$

Now, subtract this element from the old element. The element to be converted into zero is called the old element row. Finally, you have

$$4 - 1 \times 2 = 2$$

$$1 - 1 \times 1 = 0$$

$$1 - 1 \times -1 = 2$$

$$1 - 1 \times 0 = 1$$

$$0 - 1 \times 1 = -1$$

$\therefore$  The improved basic feasible solution is given in the following simplex table.



**First iteration**

	$C_j$	3	2	0	0		
$C_B$	$B$	$X_B$	$X_1$	$X_2$	$S_1$	$S_2$	Min $\frac{X_B}{X_2}$
$\leftarrow 0$	$S_1$	2	0	2	1	-1	$2/2 = 1$
3	$X_1$	2	1	-1	0	1	-
	$Z_j$	6	3	-3	0	0	
	$Z_j - C_j$		0	-5 $\uparrow$	0	0	

Since,  $Z_2 - C_2$  is the most negative,  $X_2$  enters the basis.

To find Min  $\left( \frac{X_B}{X_{i2}}, X_{i2} > 0 \right)$

$$\text{Min} \left( \frac{2}{2} \right)$$

This gives the outgoing variables. Convert the leaving element into one. This is done by dividing all the elements in the key row by 2. The remaining elements are converted to zero by using the following formula.

Here,  $-\frac{1}{2}$  is the common ratio. Put this ratio 5 times and multiply each ratio by the key row element.

$$-\frac{1}{2} \times 2$$

$$-\frac{1}{2} \times 0$$

$$-\frac{1}{2} \times 2$$

$$-1/2 \times 1$$

$$-1/2 \times -1$$

Subtract this from the old element. All the row elements which are converted into zero are called the old elements.

$$2 - \left( -\frac{1}{2} \times 2 \right) = 3$$

$$1 - (-1/2 \times 0) = 1$$

$$-1 - (-1/2 \times 2) = 0$$

$$0 - (-1/2 \times 1) = 1/2$$

$$1 - (-1/2 \times 1) = 1/2$$

**NOTES**

**NOTES**

**Second iteration**

		$C_j$	3	2	0	0
$C_B$	$B$	$X_B$	$X_1$	$X_2$	$S_1$	$S_2$
2	$X_2$	1	0	1	1/2	-1/2
3	$X_1$	3	1	0	1/2	1/2
	$Z_j$	11	3	2	5/2	1/2
	$Z_j - C_j$		0	0	5/2	1/2

Since all  $Z_j - C_j \geq 0$ , the solution is optimum. The optimal solution is Max  $Z = 11$ ,  $X_1 = 3$  and  $X_2 = 1$ .

**5.4 CONCEPT OF DUALITY AND SENSITIVITY ANALYSIS**

Every LPP (called the primal) is associated with another LPP (called its dual). Either of the problem can be considered as primal with the other as dual.

Two reasons are attributed to the importance of the concept of duality:

- (i) If a large number of constraints and a lesser number of variables constitute the primal, then the procedure for computation can be minimized by converting the primal into its dual and then finding its solution.
- (ii) While taking future decisions regarding the activities being programmed, the interpretation of the dual variables from the cost or economic point of view proves extremely useful.

**5.4.1 Formulation of a Dual Problem**

In the formulation of a dual problem, it should be first converted into its canonical form. Formulation of a dual problem involves the following modifications:

- (i) The objective function of minimization in the primal should be converted into that of maximization in the dual and vice versa.
- (ii) The number of constraints in the dual will be equal to the number of variables in its primal and vice versa.
- (iii) The right-hand side constraints in the dual will be constituted by the cost coefficients  $C_1, C_2, \dots, C_n$  in the objective function of the primal and vice versa.
- (iv) While forming the constraints for the dual, the transpose of the body matrix of the primal problem should be considered.
- (v) The variables in both problems should be positive, that is, there should be no negative values.

(vi) If the variable in the primal is unrestricted in sign, then the corresponding constraint in the dual will be an equation and vice versa.

### 5.4.2 Definition of a Dual Problem

Let the primal problem be

$$\begin{aligned} \text{Max } Z &= C_1x_1 + C_2x_2 + \dots + C_nx_n \\ \text{Subject to } & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m \\ & x_1, x_2, \dots, x_n \geq 0 \end{aligned}$$

**Dual:** The dual problem is defined as

$$\begin{aligned} \text{Min } Z' &= b_1w_1 + b_2w_2 + \dots + b_mw_m \\ \text{Subject to } & a_{11}w_1 + a_{21}w_2 + \dots + a_{m1}w_m \geq C_1 \\ & a_{12}w_1 + a_{22}w_2 + \dots + a_{m2}w_m \geq C_2 \\ & \vdots \\ & a_{1n}w_1 + a_{2n}w_2 + \dots + a_{mn}w_m \geq C_n \\ & w_1, w_2, \dots, w_m \geq 0 \end{aligned}$$

Here,  $w_1, w_2, w_3, \dots, w_m$  are called dual variables.

**Example 5.5:** Write the dual of the following primal LP problem.

$$\begin{aligned} \text{Max } Z &= x_1 + 2x_2 + x_3 \\ \text{Subject to } & 2x_1 + x_2 - x_3 \leq 2 \\ & -2x_1 + x_2 - 5x_3 \geq -6 \\ & 4x_1 + x_2 + x_3 \leq 6 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

**Solution:** Since the problem is not in the canonical form, we interchange the inequality of the second constraint.

$$\begin{aligned} \text{Max } Z &= x_1 + 2x_2 + x_3 \\ \text{Subject to } & 2x_1 + x_2 - x_3 \leq 2 \\ & 2x_1 - x_2 + 5x_3 \leq 6 \\ & 4x_1 + x_2 + x_3 \leq 6 \\ & \text{and } x_1, x_2, x_3 \geq 0 \end{aligned}$$

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**NOTES**

**Dual:** Let  $w_1, w_2, w_3$  be the dual variables.

$$\text{Min } Z' = 2w_1 + 6w_2 + 6w_3$$

$$\begin{aligned} \text{Subject to } \quad & 2w_1 + 2w_2 + 4w_3 \geq 1 \\ & +w_1 - w_2 + w_3 \geq 2 \\ & -w_1 + 5w_2 + w_3 \geq 1 \\ & w_1, w_2, w_3 \geq 0 \end{aligned}$$

**Example 5.6:** Find the dual of the following LPP.

$$\text{Max } Z = 3x_1 - x_2 + x_3$$

$$\begin{aligned} \text{Subject to } \quad & 4x_1 - x_2 \leq 8 \\ & 8x_1 + x_2 + 3x_3 \geq 12 \\ & 5x_1 - 6x_3 \leq 13 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

**Solution:** Since the problem is not in the canonical form, we interchange the inequality of the second constraint.

$$\text{Max } Z = 3x_1 - x_2 + x_3$$

$$\begin{aligned} \text{Subject to } \quad & 4x_1 - x_2 + 0x_3 \leq 8 \\ & -8x_1 - x_2 - 3x_3 \leq -12 \\ & 5x_1 + 0x_2 - 6x_3 \leq 13 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

$$\text{Max } Z = Cx$$

$$\begin{aligned} \text{Subject to } \quad & Ax \leq B \\ & x \geq 0 \end{aligned}$$

$$C = (3 \ -1) \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad b = \begin{pmatrix} 8 \\ -12 \\ 13 \end{pmatrix}$$

$$A = \begin{pmatrix} 4 & -1 & 0 \\ -8 & -1 & -3 \\ 5 & 0 & -6 \end{pmatrix}$$

**Dual:** Let  $w_1, w_2, w_3$  be the dual variables. The dual problem is

$$\text{Min } Z' = b^T W$$

$$\text{Subject to } A^T W \geq C^T \text{ and } W \geq 0$$

$$\text{i.e., Min } Z' = (8 \quad -12 \quad 13) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

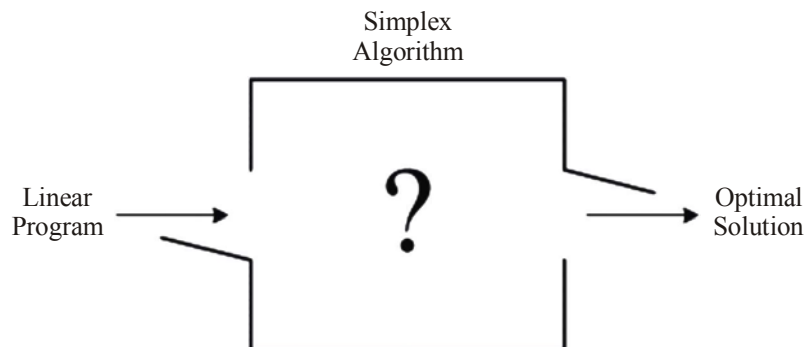
$$\text{Subject to } \begin{pmatrix} 4 & -8 & 5 \\ -1 & -1 & 0 \\ 0 & -3 & 6 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \geq \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

$$\text{Min } Z' = 8w_1 - 12w_2 + 13w_3$$

$$\begin{aligned} \text{Subject to } & 4w_1 - 8w_2 + 5w_3 \geq 3 \\ & -w_1 - w_2 + 0w_3 \geq -1 \\ & 0w_1 - 3w_2 + 6w_3 \geq 1 \\ & w_1, w_2, w_3 \geq 0 \end{aligned}$$

### 5.4.3 Sensitivity Analysis

The optimal solution of a linear programming problem is formulated using various methods. You have learned the use and importance of dual variables to solve an LPP. Here, you will learn how sensitivity analysis helps to solve repeatedly the real problem in a little different form. Generally, these scenarios crop up as an end result of parameter changes due to the involvement of new advanced technologies and the accessibility of well-organized latest information for key (input) parameters or the ‘ what-if ’ questions. Thus, sensitivity analysis helps to produce optimal solution of simple perturbations for the key parameters. For optimal solutions, consider the simplex algorithm as a ‘ black box ’ which accepts the input key parameters to solve LPP as shown below:



### NOTES

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**Example 5.7:** Illustrate sensitivity analysis using simplex method to solve the following LPP:

$$\text{Maximize } Z = 20x_1 + 10x_2$$

$$\text{Subject to, } x_1 + x_2 \leq 3$$

$$3x_1 + x_2 \leq 7$$

$$\text{And } x_1, x_2 \geq 0$$

**Solution:** Sensitivity analysis is done after making the initial and final tableau using the simplex method. Add slack variables to convert it into equation form.

$$\text{Maximize } Z = 20x_1 + 10x_2 + 0S_1 + 0S_2$$

$$\text{Subject to, } x_1 + x_2 + S_1 + 0S_2 = 3$$

$$3x_1 + x_2 + 0S_1 + S_2 = 7$$

$$\text{where } x_1, x_2 \geq 0$$

To find basic feasible solution we put  $x_1 = 0$  and  $x_2 = 0$ . This gives  $Z = 0$ ,  $S_1 = 3$  and  $S_2 = 7$ . The initial table will be as follows:

**Initial table**

		$C_j$	20	10	0	0	
$C_B$	$B$	$X_B$	$X_1$	$X_2$	$S_1$	$S_2$	$\text{Min } \frac{X_B}{X_i}$
0	$S_1$	3	1	1	1	0	$3/1 = 3$
$\leftarrow 0$	$S_2$	7	③	1	0	1	$7/3 = 2.33$
	$Z_j$	0	0	0	0	0	
	$Z_j - C_j$		$-20 \uparrow$	$-10$	0	0	

Find  $\frac{X_B}{X_i}$  for each row and also find minimum for the second row. Here,  $Z_j - C_j$  is maximum negative ( $-20$ ). Hence,  $x_1$  enters the basis and  $S_2$  leaves the basis. It is shown with the help of arrows.

Key element is 3, key row is second row and key column is  $x_1$ . Now convert the key element into entering key by dividing each element of the key row by key element using the following formula:

$$\text{New element} = \text{Old element}$$

$$-\left[ \frac{\text{Product of elements in the key row and key column}}{\text{Key element}} \right]$$

The following is the first iteration table:

		$C_j$	20	10	0	0	
$C_B$	$B$	$X_B$	$X_1$	$X_2$	$S_1$	$S_2$	
$\leftarrow 0$	$S_1$	2/3	0	(2/3)	1	-1/3	$\frac{2}{3} / \frac{2}{3} = 1$
20	$X_1$	7/3	1	1/3	0	1	$\frac{7}{3} / \frac{1}{3} = 7$
$Z_j$		140/3	20	20/3	0	20	
$Z_j - C_j$		-	0	-10/3 $\uparrow$	0	20	

Since  $Z_j - C_j$  has one value less than zero, i.e., negative value, hence this is not yet the optima solution. Value  $-10/3$  is negative, hence  $x_2$  enters the basis and  $S_1$  leaves the basis. Key row is upper row.

		$C_j$	20	10	0	0
$C_B$	$B$	$X_B$	$X_1$	$X_2$	$S_1$	$S_2$
10	$X_2$	1	0	1	3/2	-1/2
20	$X_1$	4/3	1	0	0	4/3
$Z_j$		110/3	20	10	0	25
$Z_j - C_j$			0	0	0	25

$Z_j - C_j \geq 0$  for all, hence optimal solution is reached, where  $x_1 = \frac{4}{3}, x_2 = 1,$

$$Z = \frac{80}{3} + 10 = \frac{110}{3}.$$

#### 5.4.4 Shadow Price

The price or value of any item is its exchange ratio, which is relative to some standard item. Thus, we may say that shadow price, also known as marginal value, of a constraint  $i$  is the change it induces in the optimal value of the objective function due to the result of any change in the value, i.e., on the right-hand side of the constraint  $i$ .

This can be formularized assuming,

$z$  = Objective function

$b_i$  = Right-handed side of constraint  $i$

$\pi^*$  = Standard price of constraint  $i$

At optimal solution

$$z^* = v^* = b^T \pi^* \text{ (Non-degenerate solution)}$$

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Under this situation, the change in the value of  $z$  per change of  $b_i$  for small changes in  $b_i$  is obtained by partially differentiating with objective function  $z$ , with respect to the right-handed side  $b_i$ , which is further illustrated as:

$$\frac{\delta z}{\delta b_i} = \pi_i^*$$

## NOTES

where,

$\pi_i^*$  = Price associated with the right-handed side.

It is this price, which was interpreted by Paul Samuelson as shadow price.

**Example 5.8:** A publishing firm can publish technology book, children book and business book. Each of these respectively give a net profit of ₹2, ₹4 and ₹3.

Three resources, authorship, filming and printing are needed. The available capacities are 60, 40 and 80 hours each, respectively in a week.

Technology book requires 3, 2 and 1 hours of each of the resources.

Children book requires 4, 1 and 3 hours of each of the resources.

Business book requires 2, 2 and 2 hours of each of the resources.

If we take 'T' units of Tech. Book, 'C' units of Children book and 'B' units of Business book be produced, the objective function is :

$$\text{Maximize : } 2T + 4C + 3B$$

$$\text{Subject to : } 3T + 4C + 2B \leq 60$$

$$2T + 1C + 2B \leq 40$$

$$1T + 3C + 2B \leq 80$$

$$T, C, B \geq 0$$

The final solution to the problem is given in Tableau – I. (Initial and improvement solutions are not given).

**Tableau – I : Final Solution**

$C_j$	Product	Qty	2	4	3	0	0	0
	Mix		T	C	B	$S_A$	$S_F$	$S_P$
4	C	$6\frac{2}{3}$	1/3	1	0	1/3	-1/3	0
3	B	$16\frac{2}{3}$	5/6	0	1	-1/6	2/3	0
0	$S_P$	$26\frac{2}{3}$	-5/3	0	0	-2/3	-1/3	1
	$Z_j$	$76\frac{2}{3}$	23/6	4	3	5/6	2/3	0
	$C_j - Z_j$	-	-11/6	0	0	-5/6	-2/3	0



$S_A$ ,  $S_F$  and  $S_P$  are slack resources as to authorship, filming and printing. The final solution tells that,  $6\frac{2}{3}$  children books and  $16\frac{2}{3}$  business books be produced, giving a total profit of ₹  $76\frac{2}{3}$ . No slack capacity in authorship and filming is available. But,  $26\frac{2}{3}$  hrs of printing is available. As a result printing capacity has no value, (see '0' in  $C_j - Z_j$  row), while authorship has a value of ₹  $5/6$  per hour and filming ₹  $2/3$  per hour. These are known as **Shadow prices**.

Sensitivity analysis is done with respect to Right Hand Side range (RHS range) and objective function coefficients.

### RHS Range

Since, excess printing capacity is there, by adding authorship and filming capacity we can make additional profit. But we cannot add unlimited authorship and/or filming capacities as this would affect the constraint equations beyond a point. In other words, for how large addition of these resources, the 'shadow price' will hold good? This is sensitivity analysis, analysing the sensitivity of shadow prices up to certain level of addition of the resources, shadow prices remain same, but after that level these prices become sensitivity and change.

Sensitivity analysis is done as follows: Let us do the sensitivity with respect to authorship. For that we reproduce quantity and  $S_A$  columns of final solution below and divide the former by the latter row-wise and get the values as follows in Tableau II.

Tableau – 2: Sensitivity with respect to Authorship

Quantity column	SA column	Quotient = Quantity / SA
$6\frac{2}{3}$	1/3	$(6\frac{2}{3}) / (1/3) = 20/3 \times 3/1 = 20$
$16\frac{2}{3}$	-1/6	$(16\frac{2}{3}) / (-1/6) = 50/3 \times -6/1 = -100$
$26\frac{2}{3}$	-2/3	$(26\frac{2}{3}) / (-2/3) = 80/3 \times -3/2 = -40$

The fastest positive quotient, here 20, is the level by which authorship resource can be reduced without altering shadow prices and least negative quotient, here 40, is the quantity by which authorship resource can be increased without altering shadow prices. So, the shadow price for authorship resource, at ₹  $5/6$ , is valid for a range of (-20) to (+40) hours of authorship hours. Since we have started with 60 hrs of authorship resource, the right hand side range for it is 40 (i.e.,  $60 - 20$ ) to 100 (i.e.,  $60 + 40$ ).

### NOTES

Tableau – 3

**NOTES**

Quantity column	$S_A$ column	Quotient = Quantity / $S_A$
$6\frac{2}{3}$	-1/3	$(6\frac{2}{3}) / (-1/3) = 20/3 \times -3/1 = -20$
$16\frac{2}{3}$	-2/3	$(16\frac{2}{3}) / (-2/3) = 50/3 \times 3/2 = 25$
$26\frac{2}{3}$	-1/3	$(26\frac{2}{3}) / (-1/3) = 80/3 \times -3/1 = -80$

We conclude that for filming capacity the range for which shadow price holds at the present level is  $15 = (40-25)$  to  $60 = (40+20)$ .

With regard to the surplus resource, here printing, the shadow of ₹ 0, will get altered only when its availability is reduced by quantity more than the slack quantity, i.e., below  $53\frac{1}{3}$  hrs  $(80 - 26\frac{2}{3})$ . For quantity  $\geq 53\frac{1}{3}$  the shadow price will be same as 0 as now.

**Sensitivity with respect to Objective Function Coefficients**

This analysis is done in two types: For variables already in solution and for variables not in the solution.

**Variable in Solution**

Here children book is in the solution. Sensitivity analysis here shows, how large or how small that the coefficient of children book (i.e., how large or how small the profit per unit of children book) could become without altering the optimal solution. To get the answer, we present the  $C_j - Z_j$  row and  $C_j$  row from the final solution and divide the former by the latter.

Tableau 4 : Sensitivity Analysis

$C_j - Z_j$	-11/6	0	0	-5/6	-2/3	0
$C_j$	1/3	1		1/3	-1/3	0
$(C_j - Z_j) / C_j$	$(-11/6) / 1/3$	0		$(-5/6) / (1/3)$	$(-2/3) / (-1/3)$	0*
=	-5.5	0		-2.5	2	0

The least positive quotient sets the limit by which profit can rise without altering optimal solution. Here it ₹2. The least negative quotient is the amount by which profit per children book go down without altering optical solution. Here it is ₹2.5, So, Range is : ₹1.5 to 6 [(4 - 2.5) and (4 + 2)].

Similarly we can find that the range for business book is ₹2 to ₹8. That is a decrease up to ₹ 1 and increase up to ₹ 5 will not change optimal solution.

### Variable not in Solution

In the final solution, technology book is not in solution. The reason is that loss from it is higher at ₹23/6 than its profit of ₹2 per book. To come into solution, profit per technology book must exceed ₹23/6. From current profit of ₹2, it must increase by ₹11/6 (i.e., ₹23/6 – ₹2), so that it can enter the solution.

### 5.4.5 Economic Interpretation: Interpreting the Solution for Decision-Making

We have often seen that shadow prices are being frequently used in the economic interpretation of the data in linear programming.

**Example 5.9:** To find the economic interpretation of shadow price under non-degeneracy, you will need to consider the linear programming to find out minimum of objective function  $z, x \geq 0$ , which is as follows:

$$\begin{aligned} -x_1 - 2x_2 - 3x_3 + x_4 &= z \\ x_1 + 2x_4 &= 6 \\ x_2 + 3x_4 &= 2 \\ x_3 - x_4 &= 1 \end{aligned}$$

Now, to get a optimal basic solution, we can calculate the numericals:

$$x_1 = 6, x_2 = 2, x_3 = 1, x_4 = 0, z = -13.$$

The optimal solution for the shadow price is:

$$\pi_1^\circ = -1, \pi_2^\circ = -2, \pi_3^\circ = -3,$$

as,  $z = b_1 \pi_1 + b_2 \pi_2 + b_3 \pi_3$ , where  $b = (6, 2, 1)$

It denotes,

$$\frac{\partial z}{\partial b_1} = \pi_1 = -1, \frac{\partial z}{\partial b_2} = \pi_2 = -2, \frac{\partial z}{\partial b_3} = \pi_3 = -3.$$

As these shadow prices and the changes take place in a non-degenerate situation, so they do not impact the small changes of  $b_i$ . Now, if this same situation is repeated in a degenerate situation, we will have to replace  $b_3 = 1$  by  $b_3 = 0$ ; thereby  $\partial z / \partial b_3^+ = -3$ , only if the change in  $b_3$  is positive. However, we need to keep in mind that if  $b_3$  is negative, then  $x_3$  will drop out of the basis and  $x_4$  transcends as the basics and the shadow price may be illustrated as;

$$\pi_1^\circ = -1, \pi_2^\circ = -2, \pi_3^\circ = \partial z / \partial b_3^- = -9.$$

Here, we see that the interpretation of the dual variables  $\pi$  and dual objective function  $v$  corresponds to column  $j$  of the primal problem. So, the goal of linear programming (Simplex method) is to determine whether there is a basic feasibility for optimal solution in the most cost-effective manner.

### NOTES

Thus, at iteration  $t$ ,  $v$  is the total cost of the objective function and this can be illustrated as:

$$v = \pi^T b = \sum_{i=1}^m \pi_i b_i$$

## NOTES

Here,  $\pi_i$  = Simplex multipliers which are associated with the basis  $B$ .

So, we may say that the prices of the problem of the dual variables are selected in such a manner, that there is maximization of the implicit indirect costs of the resources that are consumed by all the activities. Whenever any basic activity is conducted, it is done at a positive level and all non-basic activities are kept at a zero level.

Hence, if the primal-dual variable system is utilized, then the slack variable is maintained at a positive level in an optimal solution and the corresponding dual variable is equal to zero.

### Check Your Progress

1. List the different types of correlations.
2. What is a scatter diagram method?
3. What do you mean by coefficient of correlation?
4. What is rank correlation coefficient?
5. Fill in the blanks with appropriate words.
  - a. \_\_\_\_\_ in economic theory is known as constant returns, which means that if the amount of the input doubles, the corresponding outputs and profits are also doubled.
  - b. The region comprising of all feasible solutions is referred to as \_\_\_\_\_ region.
  - c. \_\_\_\_\_ are the non-negative variables which are added in the LHS of the constraint to convert the inequality '?' into an equation.
  - d. \_\_\_\_\_ states that there will be no feasible solution for a problem if either problem in an LPP has an unbounded solution.
6. State whether the following statements are true or false.
  - a. In the canonical form, irrespective of the objective function, all the constraints are expressed as equations and the RHS of each constraint and all variables are non-negative.
  - b. In a non-degenerate basic feasible solution one or more basic variables are zero.
  - c. In a dual, if one is a minimization problem, then the other will be a maximization problem.
  - d. Simplex method is an iterative procedure used for solving an LPP in a finite number of steps.

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## 5.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

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1. The types of correlations are:
  - (a) Positive or negative correlations
  - (b) Linear or non-linear correlations
  - (c) Simple, partial or multiple correlations
2. Scatter diagram is a method to calculate the constants in regression models that makes use of scatter diagram or dot diagram. A scatter diagram is a diagram that represents two series with the known variables, i.e., independent variable plotted on the  $X$ -axis and the variable to be estimated, i.e., dependent variable to be plotted on the  $Y$ -axis.
3. The coefficient of correlation, which is symbolically denoted by  $r$ , is an important measure to describe how well one variable explains another. It measures the degree of relationship between two causally-related variables. The value of this coefficient can never be more than  $+1$  or  $-1$ . Thus,  $+1$  and  $-1$  are the limits of this coefficient.
4. The rank correlation, written  $r$ , is a descriptive index of agreement between ranks over individuals. It is the same as the ordinary coefficient of correlation computed on ranks, but its formula is simpler.
5.
  - a. Linearity
  - b. Feasible
  - c. Slack variables
  - d. Existence Theorem
6.
  - a. False
  - b. False
  - c. True
  - d. True

## NOTES

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## 5.6 SUMMARY

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- A set of values  $X_1, X_2, \dots, X_n$ , which satisfies the constraints of the LPP is called its solution.
- An LP problem can have either a unique optimal solution or an infinite number of optimal solutions or an unbounded solution or it can have no solution.
- There are two types of basic feasible solutions, namely degenerate and non-degenerate.

## NOTES

- In the standard form of LPP, all constraints are expressed as equations; while in the standard form all constraints are expressed as equations by introducing additional variables called slack variables and surplus variables.
- Simplex method is an iterative procedure used for solving an LPP in a finite number of steps.
- Every LPP (called the primal) is associated with another LPP (called its dual).

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### 5.7 KEY WORDS

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- **Simplex method:** The simplex method is an iterative procedure for solving an LPP in a finite number of steps. This method provides an algorithm which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more as the case may be at the previous vertex

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### 5.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

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#### Short Answer Questions

1. What are slack variables? Where are they used?
2. What is the simplex method?
3. Does every LPP solution have an optimal solution? Why?

#### Long Answer Questions

1. Solve the following by graphical method:

(i)  $\text{Max } Z = X_1 - 3X_2$   
Subject to,  $X_1 + X_2 \leq 300$   
 $X_1 - 2X_2 \leq 200$   
 $2X_1 + X_2 \leq 100$   
 $X_2 \leq 200$   
 $X_1, X_2 \geq 0$

(ii)  $\text{Max } Z = 5X + 8Y$   
Subject to,  $3X + 2Y \leq 36$   
 $X + 2Y \leq 20$   
 $3X + 4Y \leq 42$   
 $X, Y \geq 0$

(iii)  $\text{Max } Z = X - 3Y$   
Subject to,  $X + Y \leq 300$   
 $X - 2Y \leq 200$

$$\begin{aligned} X + Y &\leq 100 \\ Y &\geq 200 \\ \text{and } X, Y &\geq 0 \end{aligned}$$

2. Solve graphically the following LPP:

$$\begin{aligned} \text{Max } Z &= 20X_1 + 10X_2 \\ \text{Subject to, } X_1 + 2X_2 &\leq 40 \\ 3X_1 + X_2 &\geq 30 \\ 4X_1 + 3X_2 &\geq 60 \\ \text{and } X_1, X_2 &\geq 0 \end{aligned}$$

3. A company produces two different products  $A$  and  $B$ . The company makes a profit of ₹ 40 and ₹ 30 per unit on  $A$  and  $B$ , respectively. The production process has a capacity of 30,000 man hours. It takes 3 hours to produce one unit of  $A$  and one hour to produce one unit of  $B$ . The market survey indicates that the maximum number of units of product  $A$  that can be sold is 8000 and those of  $B$  is 12000 units. Formulate the problem and solve it by graphical method to get maximum profit.

4. Using simplex method, find non-negative values of  $X_1, X_2$  and  $X_3$  when

(i)  $\text{Max } Z = X_1 + 4X_2 + 5X_3$

Subject to the constraints,

$$3X_1 + 6X_2 + 3X_3 \leq 22$$

$$X_1 + 2X_2 + 3X_3 \leq 14$$

$$3X_1 + 2X_2 \leq 14$$

$$X_1, X_2, X_3 \geq 0$$

(ii)  $\text{Max } Z = X_1 + X_2 + 3X_3$

Subject to,  $3X_1 + 2X_2 + X_3 \leq 2$

$$2X_1 + X_2 + 2X_3 \leq 2$$

$$X_1, X_2, X_3 \geq 0$$

(iii)  $\text{Max } Z = 10X_1 + 6X_2$

Subject to,  $X_1 + X_2 \leq 2$

$$2X_1 + X_2 \leq 4$$

$$3X_1 + 8X_2 \leq 12$$

$$X_1, X_2 \geq 0$$

(iv)  $\text{Max } Z = 30X_1 + 23X_2 + 29X_3$

Subject to the constraints,

$$6X_1 + 5X_2 + 3X_3 \leq 52$$

$$6X_1 + 2X_2 + 5X_3 \leq 14$$

$$X_1, X_2, X_3 \geq 0$$

## NOTES

## NOTES

$$(v) \text{ Max } Z = X_1 + 2X_2 + X_3$$

$$\text{Subject to, } 2X_1 + X_2 - X_3 \geq -2$$

$$-2X_1 + X_2 - 5X_3 \leq 6$$

$$4X_1 + X_2 + X_3 \leq 6$$

$$X_1, X_2, X_3 \geq 0$$

5. A manufacturer is engaged in producing 2 products  $X$  and  $Y$ , the contribution margin being ₹ 15 and ₹ 45 respectively.  $A$  unit of product  $X$  requires 1 unit of facility  $A$  and 0.5 unit of facility  $B$ .  $A$  unit of product  $Y$  requires 1.6 units of facility  $A$ , 2.0 units of facility  $B$  and 1 unit of raw material  $C$ . The availability of total facility  $A$ ,  $B$  and raw material  $C$  during a particular time period are 240, 162 and 50 units respectively.

Find out the product-mix which will maximize the contribution margin by simplex method.

6. A firm has available 240, 370 and 180 kg of wood, plastic and steel respectively. The firm produces two products  $A$  and  $B$ . Each unit of  $A$  requires 1, 3 and 2 kg of wood, plastic and steel, respectively. The corresponding requirement for each unit of  $B$  are 3, 4 and 1, respectively. If  $A$  sells for ₹ 4 and  $B$  for ₹ 6, determine how many units of  $A$  and  $B$  should be produced in order to obtain the maximum gross income. Use the simplex method.

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## 5.9 FURTHER READINGS

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## UNIT 6 SPECIAL ALGORITHMS OF LPP

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### NOTES

#### Structure

- 6.0 Introduction
- 6.1 Objectives
- 6.2 Basics of Transportation Problem
  - 6.2.1 Transportation Algorithm: Balance and Unbalanced Problem
- 6.3 Assignment and Travelling Executive Algorithms
  - 6.3.1 The Travelling Salesman Problem
  - 6.3.2 Mathematical Formulation
- 6.4 Formulation and Solving Methods
  - 6.4.1 North West Corner Rule
  - 6.4.2 Least Cost or Matrix Minima Method
  - 6.4.3 Vogel's Approximation Method (VAM)
- 6.5 MODI Method
- 6.6 Answers to Check Your Progress Questions
- 6.7 Summary
- 6.8 Key Words
- 6.9 Self Assessment Questions and Exercises
- 6.10 Further Readings

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### 6.0 INTRODUCTION

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This unit will discuss the transportation problem, which is a subclass of a Linear Programming Problem (LPP). Transportation problems deal with the objective of transporting various quantities of a single homogeneous commodity initially stored at various origins, to different destinations, in a way that keeps transportation cost at a minimum.

You will learn about applications of the transportation problem and rules to solve such problems. The solution of any transportation problem is obtained in two stages, initial solution and optimal solution. There are three methods of obtaining an initial solution. These are: North West Corner Rule, Least Cost Method and Vogel's Approximation Method (VAM). VAM is preferred since the solution obtained this way is very close to the optimal solution. The optimal solution of any transportation problem is a feasible solution that minimizes the total cost. An optimal solution is the second stage of a solution obtained by improving the initial solution. The MODI method is used to obtain optimal solutions and optimality tests.

**NOTES**

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## 6.1 OBJECTIVES

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After going through this unit, you will be able to:

- Know about the transportation problem
- Explain the three methods of finding an initial solution
- Use the MODI method for finding optimal solution
- Analyse assignment and travelling executive algorithms

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## 6.2 BASICS OF TRANSPORTATION PROBLEM

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The transportation problem is one of the subclasses of LPP (Linear Programming Problem) in which the objective is to transport various quantities of a single homogeneous commodity that are initially stored at various origins to different destinations in such a way that the transportation cost is minimum. To achieve this objective we must know the amount and location of available supplies and the quantities demanded. In addition, we must know the costs that result from transporting one unit of commodity from various origins to various destinations.

### Elementary Transportation Problem

Consider a transportation problem with  $m$  origins (rows) and  $n$  destinations (columns). Let  $C_{ij}$  be the cost of transporting one unit of the product from the  $i$ th origin to  $j$ th destination,  $a_i$  the quantity of commodity available at origin  $i$ ,  $b_j$  the quantity of commodity needed at destination  $j$ .  $X_{ij}$  is the quantity transported from  $i$ th origin to  $j$ th destination. This transportation problem can be stated in the following tabular form.

		<i>Destinations</i>					Capacity
		1	2	3	...	$n$	
<i>Origins</i>	1	$C_{11}$ $X_{11}$	$C_{12}$ $X_{12}$	$C_{13}$ $X_{13}$	...	$C_{1n}$ $X_{1n}$	$a_1$
	2	$C_{21}$ $X_{21}$	$C_{22}$ $X_{22}$	$C_{23}$ $X_{23}$	...	$C_{2n}$ $X_{2n}$	$a_2$
	3	$C_{31}$ $X_{31}$	$C_{32}$ $X_{32}$	$C_{33}$ $X_{33}$	...	$C_{3n}$ $X_{3n}$	$a_3$
	$m$	$C_{m1}$ $X_{m1}$	$C_{m2}$ $X_{m2}$	$C_{m3}$ $X_{m3}$	...	$C_{mn}$ $X_{mn}$	$a_m$
	<i>Demand</i>	$b_1$	$b_2$	$b_3$	...	$b_n$	$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$

The linear programming model representing the transportation problem is given by,

$$\text{Minimize } Z = \sum_{i=1}^m \sum_{j=1}^n C_{ij} X_{ij}$$

Subject to the constraints,

$$\sum_{j=1}^n X_{ij} = a_i \quad i = 1, 2, \dots, n \text{ (Row Sum)}$$

$$\sum_{i=1}^m X_{ij} = b_j \quad j = 1, 2, \dots, n \text{ (Column Sum)}$$

$$X_{ij} \geq 0 \quad \text{For all } i \text{ and } j$$

The given transportation problem is said to be balanced if,

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$$

i.e., the total supply is equal to the total demand.

### Definitions

**Feasible Solution:** Any set of non-negative allocations ( $X_{ij} > 0$ ) which satisfies the row and column sum (rim requirement) is called a feasible solution.

**Basic Feasible Solution:** A feasible solution is called a basic feasible solution if the number of non-negative allocations is equal to  $m + n - 1$ , where  $m$  is the number of rows and  $n$  the number of columns in a transportation table.

**Non-degenerate Basic Feasible Solution:** Any feasible solution to a transportation problem containing  $m$  origins and  $n$  destinations is said to be non-degenerate, if it contains  $m + n - 1$  occupied cells and each allocation is in independent positions.

The allocations are said to be in independent positions if it is impossible to form a closed path. Closed path means by allowing horizontal and vertical lines and when all the corner cells are occupied.

The allocations in the following tables are not in independent positions.

	*	*
	*	*

*		*
*		*

	*	*	
	*		
	*	*	

### NOTES

The allocations in the following tables are in independent positions.

	*	
*	*	*
*		

*	*		
	*		*
		*	*

## NOTES

**Degenerate Basic Feasible Solution:** If a basic feasible solution contains less than  $m + n - 1$  non-negative allocations then it is said to be degenerate.

### 6.2.1 Transportation Algorithm: Balance and Unbalanced Problem

This algorithm can be used for minimizing the transportation cost for goods from  $O$  origins to  $D$  destinations and there may be  $O \times D$  number of direct routes from  $O$  origins to  $D$  destinations. Problem is balanced when sum of supplies at  $O$  sources is equal to sum of demands at  $D$  destinations. If it is not so, then this problem is not balanced. There may be two such situations. Supply may be lesser than demand and in that case it is balanced by adding dummy supply node. If demand is lesser than supply then dummy demand node is added to make it a balanced problem. Thus, before starting to use this algorithm, problem should be made balanced, if it is not balanced. The algorithm for balanced problem is being described below.

Data is presented in tabular form. As a convention, origins are put on left side of the table with quantity to be supplied listed towards right side and demands are put on top with quantity of demand towards the bottom side. Unit cost of transportation is put at the top of every cell within a small box. Zero unit cost shows unshipped unit column in case supply is in excess of the demand. Similarly, a unit cost either penalty or zero shows shortage row in supplies that are lesser than demand.

The algorithm has two phases. In phase I this makes allocation for supplies on demands by making utilizing an approach of minimum unit cost for generating a feasible solution. This feasible solution may not be optimal. Optimization is done in second phase and in this phase checking is done for optimality conditions and improvement is done for reducing the cost if optimality conditions are not satisfied. This second phase adopts iterative steps and stops only when optimality conditions are satisfied. Once done, no further steps are required.

### Basic and Non-basic Cells

Basic cells are those that indicate positive values and non-basic cells have zero value for flow. According to transportation problem number of basic cells will be exactly  $m + n - 1$ .

Algorithm follows as below:

**Step 0**

**Initialization:** Before starting to solve the problem, it should be balanced. If not then make it balanced by ‘unshipped supply’ column in case demand is less than supply or by adding ‘shortage’ row in case supply is less than the supply. Put zero for unit costs in the column for unshipped supply. Put either penalty costs or zero in a row that shows shortage.

**Phase I: To find initial ‘feasible solution’**

**Step 1:** Locate cell with minimum cost having positive supply as well as demand, then make allocation in that cell having residuals as minimum.

**Step 2:** Reduce residual supply/demand as per allocations made above (Step 1). Do this till all demands are met and then proceed to Phase II. If all demands are not met go to Step 1.

**Phase II: Optimal Solution**

Carry out check for Optimality.

**Step 3:** For rows and columns find dual values,  $u_i$  for rows and  $v_j$  for columns. For this, set  $u_1$  (first dual value) to 0 followed by solution of triangular dual equations one by one. These **dual equations** are to be applied for basic cells only as follows,

$$C_{ij} = u_i + v_j$$

Where,  $C_{ij}$  denotes unit cost, as given for that cell and either  $v_j$  or  $u_i$  is already known. Here,  $v_j$  denotes dual value of column  $j$  and  $u_i$  denotes that for row  $i$ . These are taken as already known. Other dual value is computed from the following equation,

$$C_{ij} = u_i + v_j.$$

Thus, dual values of every cell can be computed by setting the first to zero and appropriate order is used for dual equations for basic cells.

**Step 4:** Optimality conditions are expressed as **reduced costs** in case of all non-basic cells as given below:

$$\Delta_{ij} = C_{ij} - (u_i + v_j)$$

Reduced cost ( $\Delta_{ij}$ ) in case of non-basic cells represent net change in unit cost resulting due to movement of cell  $ij$  to solve and adjusting around a cycle that has been created in this way for basic cells to find current solution. This is shown in Step 5. Hence if one  $\Delta_{ij}$  is positive, then by use of this cell total transportation will increase, but when  $\Delta_{ij}$  is negative for a cell, this will cause reduction in total transportation. If all reduced costs ( $\Delta_{ij}$ ) are positive, i.e.,  $\Delta_{ij} > 0$ , **optimality conditions** are satisfied and no further improvement is possible and algorithm terminates at this point. But if at least one  $\Delta_{ij}$  is negative, optimality conditions are not satisfied and there is possibility of reduction in costs. If optimality conditions are not satisfied algorithm continues as given in Step 5.

**NOTES**

## NOTES

### Adjustment for reducing cost

**Step 5:** Select a cell 'ij' that is most negative for  $\Delta_{ij}$ . This becomes entering variable. Put (+) sign for identifying it. To maintain constraints of balance in supply and demand, locate basic cells for  $i$ th row as well as  $j$ th column that compensates for increase in value in cell 'ij'. Put negative (-) sign in these cells. This process should be continued to get one cycle in which (+) and (-) are marked. Such a cycle will be unique and if 'dead ends' are encountered in such a process, make a back track and one amongst other alternatives are tried. A cycle has rows/column having non-basic cells for holding compensating (+) or (-) sign. This may require trial and error for finding it.

**Step 6:** After determining every basic cell within this cycle, adjustment is obtained as minimum value in basic cells that are negative. This is known as adjustment amount and let it be called 'aa'. Add this to every cell value that is positive and marked with (+) sign and subsequently deduct this from cells having negative (-) sign and then drop those from the basis that becomes zero. In case two or more cells become zero from such adjustment, drop only one of these that have greatest  $C_{ij}$  value. This is necessary for maintaining basic cells having  $m+n-1$  number for computing dual values. Reduction in cost that is associated with such a change is found as product of reduced cost  $\Delta_{ij}$  for incoming cell multiplied by cell value previously held by outgoing cell.

After finding this new solution, move to the third Step 3 for checking conditions of optimality. If optimality condition is satisfied, algorithm terminates.

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## 6.3 ASSIGNMENT AND TRAVELLING EXECUTIVE ALGORITHMS

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Let us analyse the assignment and travelling executive algorithms.

### 6.3.1 The Travelling Salesman Problem

Assume that a salesman has to visit  $n$  cities. He wishes to start from a particular city, visits each city once and then returns to his starting point. His objective is to select the sequence in which the cities are visited in such a way that his total travelling time is minimized.

To visit 2 cities A and B, there is no choice. To visit 3 cities we have 2! possible routes. For 4 cities we have 3! possible routes. In general to visit  $n$  cities there are  $(n-1)!$  possible routes.

### 6.3.2 Mathematical Formulation

Let  $C_{ij}$  be the distance or time or cost of going from city  $i$  to city  $j$ . The decision variable  $X_{ij}$  be 1 if the salesman travels from city  $i$  to city  $j$  and otherwise 0.

The objective is to minimize the travelling time.

$$Z = \sum_{i=1}^n \sum_{j=1}^n C_{ij} X_{ij}$$

Subject to the constraints,

$$\sum_{j=1}^n X_{ij} = 1, i = 2, \dots, n$$

$$\sum_{i=1}^n X_{ij} = 1, j = 2, \dots, n$$

Subject to the additional constraint that  $X_{ij}$  is so chosen that no city is visited twice before all the cities are completely visited.

In particular, going directly from  $i$  to  $i$  is not permitted. Which means  $C_{ij} = \infty$ , when  $i = j$ .

In travelling salesman problem we cannot choose the element along the diagonal and this can be avoided by filling the diagonal with infinitely large elements.

The travelling salesman problem is very similar to the assignment problem except that in the former case, there is an additional restriction that  $X_{ij}$  is so chosen that no city is visited twice before the tour of all the cities is completed.

Treat the problem as an assignment problem and solve it using the same procedures. If the optimal solution of the assignment problem satisfies the additional constraint, then it is also an optimal solution of the given travelling salesman problem. If the solution to the assignment problem does not satisfy the additional restriction then after solving the problem by assignment technique we use the method of enumeration.

**Example 6.1:** A travelling salesman has to visit 5 cities. He wishes to start from a particular city, visit each city once and then return to his starting point. Cost of going from one city to another is shown below. You are required to find the least cost route.

		To City				
		A	B	C	D	E
From City	A	$\infty$	4	10	14	2
	B	12	$\infty$	6	10	4
	C	16	14	$\infty$	8	14
	D	24	8	12	$\infty$	10
	E	2	6	4	16	$\infty$

**Solution:** First we solve this problem as an assignment problem.

Subtract the minimum element in each row from all the elements in its row.

		A	B	C	D	E
A	$\infty$	2	8	12	0	
B	8	$\infty$	2	6	0	
C	8	6	$\infty$	0	6	
D	16	0	4	$\infty$	2	
E	0	4	2	14	$\infty$	

## NOTES

**NOTES**

Subtract the minimum element in each column from all the elements in its column.

$$\begin{matrix} & A & B & C & D & E \\ A & \infty & 2 & 8 & 12 & 0 \\ B & 8 & \infty & 2 & 6 & 0 \\ C & 8 & 6 & \infty & 0 & 6 \\ D & 16 & 0 & 4 & \infty & 2 \\ E & 0 & 4 & 2 & 14 & \infty \end{matrix}$$

We have the first modified matrix. Draw minimum number of lines to cover all zeros.

$$\begin{matrix} & A & B & C & D & E \\ A & \infty & 2 & 8 & 12 & 0 \\ B & 8 & \infty & 2 & 6 & 0 \\ C & 8 & 6 & \infty & 0 & 6 \\ D & 16 & 0 & 4 & \infty & 2 \\ E & 0 & 4 & 2 & 14 & \infty \end{matrix}$$

Here  $N = 4 < n = 5$ , i.e.,  $N < n$ . Subtract the smallest uncovered element from all the uncovered elements and add to the element which is in the point of intersection of lines. Hence, we get the second modified matrix.

$$\begin{matrix} & A & B & C & D & E \\ A & \infty & 0 & 6 & 12 & 0 \\ B & 8 & \infty & 0 & 6 & 0 \\ C & 8 & 4 & \infty & 0 & 6 \\ D & 18 & 0 & 4 & \infty & 4 \\ E & 0 & 2 & 0 & 14 & \infty \end{matrix}$$

$N = 5 = n = 5 =$  Order of matrix. We make assignment.

**Assignment**

$$\begin{matrix} & A & B & C & D & E \\ A & \infty & \times & 6 & 12 & \textcircled{0} & A \rightarrow E \\ B & 8 & \infty & \textcircled{0} & 6 & \times & B \rightarrow C \\ C & 8 & 4 & \infty & \textcircled{0} & 6 & C \rightarrow D \\ D & 18 & \textcircled{0} & 4 & \infty & 4 & D \rightarrow B \\ E & \textcircled{0} & 2 & \times & 14 & \infty & E \rightarrow A \\ & & & & & & A \rightarrow E \rightarrow A \end{matrix}$$

As the salesman should go from  $A$  to  $E$  and then come back to  $A$  without covering  $B, C, D$  which is contradicting the fact that no city is visited twice before all the cities are visited.

Hence, we obtain the next best solution by bringing the next minimum non-zero element namely 4.

$$\begin{matrix} & A & B & C & D & E \\ A & \infty & \textcircled{0} & 6 & 12 & 0 \\ B & 8 & \infty & \textcircled{0} & 6 & \times \\ C & 8 & 4 & \infty & \textcircled{0} & 6 \\ D & 18 & \times & 4 & \infty & \textcircled{4} \\ E & \textcircled{0} & 2 & \times & 14 & \infty \end{matrix}$$



$$A \rightarrow B, B \rightarrow C, C \rightarrow D, D \rightarrow E, E \rightarrow A$$

Since all the cities have been visited and no city is visited twice before completing the tour of all the cities, we have an optimal solution to the travelling salesman.

The least cost route is  $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow A$ .

$$\text{Total Cost} = 4 + 6 + 8 + 10 + 2 = ₹ 30.$$

**Example 6.2:** A machine operator processes five types of items on his machine each week and must choose a sequence for them. The set-up cost per change depends on the items presently on the machine and the set-up to be made according to the following table.

		To Item				
		A	B	C	D	E
From Item	A	$\infty$	4	7	3	4
	B	4	$\infty$	6	3	4
	C	7	6	$\infty$	7	5
	D	3	3	7	$\infty$	7
	E	4	4	5	7	$\infty$

If he processes each type of item once and only once in each week, how should he sequence the items on his machine in order to minimize the total set-up cost?

**Solution:** Reduce the cost matrix and make assignments in rows and columns having single row.

Modify the matrix by subtracting the least element from all the elements in its row and also in its column.

		A	B	C	D	E
A	$\infty$	1	4	0	1	
B	1	$\infty$	3	0	1	
C	2	1	$\infty$	2	0	
D	0	0	4	$\infty$	4	
E	0	0	1	3	$\infty$	

		A	B	C	D	E
A	$\infty$	1	4	0	1	
B	1	$\infty$	3	0	1	
C	2	1	$\infty$	2	0	
D	0	0	4	$\infty$	4	
E	0	0	1	3	$\infty$	

Here,  $N = 4 < n = 5$ , i.e.,  $N < n$ .

Subtract the smallest uncovered element from all the uncovered elements and add to the element which is at the point of intersection of lines and get the reduced second modified matrix.

## NOTES

**NOTES**

$$\begin{array}{c}
 A \quad B \quad C \quad D \quad E \\
 \begin{array}{l}
 A \left[ \begin{array}{ccccc}
 \infty & 0 & 3 & 0 & 1 \\
 0 & \infty & 2 & 0 & 1 \\
 1 & 0 & \infty & 1 & 0 \\
 0 & 0 & 4 & \infty & 5 \\
 0 & 0 & 1 & 4 & \infty
 \end{array} \right] \\
 B \\
 C \\
 D \\
 E
 \end{array}
 \end{array}$$

Here,  $N = 5 = n = 5 =$  Order of matrix. We make the assignment.

**Assignment**

We make a solution by considering the next smallest non-zero element by considering 1.

$$\begin{array}{c}
 A \quad B \quad C \quad D \quad E \\
 \begin{array}{l}
 A \left[ \begin{array}{ccccc}
 \infty & \infty & 3 & \infty & \textcircled{1} \\
 \infty & \infty & 2 & \textcircled{1} & 1 \\
 1 & \textcircled{1} & \infty & 1 & \infty \\
 \textcircled{1} & \infty & 4 & \infty & 5 \\
 \infty & \infty & \textcircled{1} & 4 & \infty
 \end{array} \right] \begin{array}{l}
 A \rightarrow E \\
 B \rightarrow D \\
 C \rightarrow B \\
 D \rightarrow A \\
 E \rightarrow C
 \end{array} \\
 B \\
 C \\
 D \\
 E
 \end{array}
 \end{array}$$

$A \rightarrow E, E \rightarrow C, C \rightarrow B, B \rightarrow D, D \rightarrow A$

We get the solution  $A \rightarrow E \rightarrow C \rightarrow B \rightarrow D \rightarrow A$ .

This schedule provides the required solution in which each item is not processed once in a week.

i.e.,  $A \rightarrow E \rightarrow C \rightarrow B \rightarrow D \rightarrow A$ .

The total set-up cost comes to ₹ 21.

**6.4 FORMULATION AND SOLVING METHODS**

Optimal solution is a feasible solution (not necessarily basic) which minimizes the total cost.

The solution of a Transportation Problem (TP) can be obtained in two stages, namely initial solution and optimum solution.

Initial solution can be obtained by using any one of the three methods, viz.

- (i) North West Corner Rule (NWCR)
- (ii) Least Cost Method or Matrix Minima Method
- (iii) Vogel's Approximation Method (VAM)

VAM is preferred over the other two methods, since the initial basic feasible solution obtained by this method is either optimal or very close to the optimal solution.

The cells in the transportation table can be classified as occupied cells and unoccupied cells. The allocated cells in the transportation table are called **occupied cells** and **empty cells** in a transportation table are called **unoccupied cells**.

The improved solution of the initial basic feasible solution is called optimal solution which is the second stage of solution, that can be obtained by MODI (MODified Distribution Method).

### 6.4.1 North West Corner Rule

**Step 1:** Starting with the cell at the upper left corner (North West) of the transportation matrix we allocate as much as possible so that either the capacity of the first row is exhausted or the destination requirement of the first column is satisfied, i.e.,  $X_{11} = \text{Min}(a_1, b_1)$ .

**Step 2:** If  $b_1 > a_1$ , we move down vertically to the second row and make the second allocation of magnitude  $X_{21} = \text{Min}(a_2, b_1 - X_{11})$  in the cell (2, 1).

If  $b_1 < a_1$ , move right horizontally to the second column and make the second allocation of magnitude  $X_{12} = \text{Min}(a_1, X_{11} - b_1)$  in the cell (1, 2).

If  $b_1 = a_1$ , there is a tie for the second allocation. We make the second allocations of magnitude,

$$X_{12} = \text{Min}(a_1 - a_1, b_1) = 0 \text{ in the cell (1, 2)}$$

$$\text{or, } X_{21} = \text{Min}(a_2, b_1 - b_1) = 0 \text{ in the cell (2, 1)}$$

**Step 3:** Repeat Steps 1 and 2 moving down towards the lower right corner of the transportation table until all the rim requirements are satisfied.

**Example 6.3:** Obtain the initial basic feasible solution of a transportation problem whose cost and rim requirement table is as follows:

Origin\Destination	$D_1$	$D_2$	$D_3$	Supply
$O_1$	2	7	4	5
$O_2$	3	3	1	8
$O_3$	5	4	7	7
$O_4$	1	6	2	14
<b>Demand</b>	7	9	18	34

**Solution:** Since  $\sum a_i = 34 = \sum b_j$ , there exists a feasible solution to the transportation problem. We obtain initial feasible solution as follows.

The first allocation is made in the cell (1, 1) the magnitude being,

$$X_{11} = \text{Min}(5, 7) = 5.$$

The second allocation is made in the cell (2, 1) and the magnitude of the allocation is given by  $X_{21} = \text{Min}(8, 7 - 5) = 2$ .

	$D_1$	$D_2$	$D_3$	Supply
$O_1$	⑤ 2	7	4	5 0
$O_2$	② 3	⑥ 3	1	8 6 0
$O_3$	5	③ 4	④ 7	7 4 0
$O_4$	1	6	⑭ 2	14 0
<b>Demand</b>	7 -2 0	9 -3 0	18 -14 0	34

### NOTES

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The third allocation is made in the cell (2, 2) the magnitude  $X_{22} = \text{Min}(8 - 2, 9) = 6$ .

The magnitude of the fourth allocation is made in the cell (3, 2) given by  $X_{32} = \text{Min}(7, 9 - 6) = 3$ .

The fifth allocation is made in the cell (3, 3) with magnitude  $X_{33} = \text{Min}(7 - 3, 14) = 4$ .

The final allocation is made in the cell (4, 3) with magnitude  $X_{43} = \text{Min}(14, 18 - 4) = 14$ .

Hence, we get the initial basic feasible solution to the given TP and is given by,

$$X_{11} = 5; X_{21} = 2; X_{22} = 6; X_{32} = 3; X_{33} = 4; X_{43} = 14$$

$$\text{Total Cost} = 2 \times 5 + 3 \times 2 + 3 \times 6 + 3 \times 4 + 4 \times 7 + 2 \times 14$$

$$= 10 + 6 + 18 + 12 + 28 + 28 = ₹ 102$$

**Example 6.4:** Determine an initial basic feasible solution to the following transportation problem using North West corner rule.

	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$O_1$	6	4	1	5	14
$O_2$	8	9	2	7	16
$O_3$	4	3	6	2	5
<b>Required</b>	6	10	15	4	35

**Solution:** The problem is a balanced TP as the total supply is equal to the total demand. Using the steps we find the initial basic feasible solution as given in the following table.

	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$O_1$	6 ⑥	4 ⑧	1	5	<del>14</del> <del>8</del> 0
$O_2$	8	9 ②	2 ⑭	7	<del>16</del> <del>14</del> 0
$O_3$	4	3	6 ①	2 ④	<del>5</del> <del>4</del>
<b>Demand</b>	<del>6</del>	<del>10</del> 2 0	<del>15</del> 1 0	<del>4</del>	35

The solution is given by,

$$X_{11} = 6; X_{12} = 8; X_{22} = 2; X_{23} = 14; X_{33} = 1; X_{34} = 4$$

$$\text{Total Cost} = 6 \times 6 + 4 \times 8 + 2 \times 9 + 2 \times 14 + 6 \times 1 + 2 \times 4$$

$$= 36 + 32 + 18 + 28 + 6 + 8 = ₹ 128.$$

**6.4.2 Least Cost or Matrix Minima Method**

**Step 1:** Determine the smallest cost in the cost matrix of the transportation table. Let it be  $C_{ij}$ . Allocate  $X_{ij} = \text{Min}(a_i, b_j)$  in the cell  $(i, j)$ .

**Step 2:** If  $X_{ij} = a_i$  cross off the  $i$ th row of the transportation table and decrease  $b_j$  by  $a_i$ . Then go to Step 3.

If  $X_{ij} = b_j$  cross off the  $j$ th column of the transportation table and decrease  $a_i$  by  $b_j$ . Go to Step 3.

If  $X_{ij} = a_i = b_j$  cross off either the  $i$ th row or the  $j$ th column but not both.

**Step 3:** Repeat Steps 1 and 2 for the resulting reduced transportation table until all the rim requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

**Example 6.5:** Obtain an initial feasible solution to the following TP using Matrix Minima Method.

	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$O_1$	1	2	3	4	6
$O_2$	4	3	2	0	8
$O_3$	0	2	2	1	10
<b>Demand</b>	4	6	8	6	24

**Solution:** Since  $\sum a_i = \sum b_j = 24$ , there exists a feasible solution to the TP using the steps in the least cost method, the first allocation is made in the cell (3, 1) the magnitude being  $X_{31} = 4$ . This satisfies the demand at the destination  $D_1$  and we delete this column from the table as it is exhausted.

	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$O_1$	<del>1</del>	⑥	3	4	<del>6</del> 0
$O_2$	4	3	②	⑥	<del>8</del> 2
$O_3$	④	2	⑥	1	<del>10</del> 6
<b>Demand</b>	<del>4</del> 0	<del>6</del> 0	<del>8</del> 2 0	<del>6</del> 0	24

The second allocation is made in the cell (2, 4) with magnitude  $X_{24} = \text{Min}(6, 8) = 6$ . Since it satisfies the demand at the destination  $D_4$ , it is deleted from the table. From the reduced table, the third allocation is made in the cell (3, 3) with magnitude  $X_{33} = \text{Min}(8, 6) = 6$ . The next allocation is made in the cell (2, 3) with magnitude  $X_{23}$  of  $\text{Min}(2, 2) = 2$ . Finally, the allocation is made in the cell (1, 2) with magnitude  $X_{12} = \text{Min}(6, 6) = 6$ . Now, all the requirements have been satisfied and hence, the initial feasible solution is obtained.

The solution is given by,

$$X_{12} = 6; X_{23} = 2; X_{24} = 6; X_{31} = 4; X_{33} = 6$$

Since the total number of occupied cells =  $5 < m + n - 1$

We get a degenerate solution.

$$\begin{aligned} \text{Total Cost} &= 6 \times 2 + 2 \times 2 + 6 \times 0 + 4 \times 0 + 6 \times 2 \\ &= 12 + 4 + 12 = ₹ 28. \end{aligned}$$

## NOTES

NOTES

**Example 6.6:** Determine an initial basic feasible solution for the following TP, using the Least Cost Method.

	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$O_1$	6	4	1	5	14
$O_2$	8	9	2	7	16
$O_3$	4	3	6	2	5
<b>Demand</b>	6	10	15	4	35

**Solution:** Since  $\sum a_i = \sum b_j$ , there exists a basic feasible solution. Using the steps in least cost method we make the first allocation to the cell (1, 3) with magnitude  $X_{13} = \text{Min}(14, 15) = 14$  (as it is the cell having the least cost).

This allocation exhausts the first row supply. Hence, the first row is deleted. From the reduced table, the next allocation is made in the next least cost cell (2, 3) which is chosen arbitrarily with magnitude  $X_{23} = \text{Min}(1, 16) = 1$ . This exhausts the 3rd column destination.

From the reduced table the next least cost cell is (3, 4) for which allocation is made with magnitude  $\text{Min}(4, 5) = 4$ . This exhausts the destination  $D_4$  requirement. Delete this 4th column from the table. The next allocation is made in the cell (3, 2) with magnitude  $X_{32} = \text{Min}(1, 10) = 1$  which exhausts the 3rd origin capacity. Hence, the 3rd row is exhausted. From the reduced table the next allocation is given to the cell (2,1) with magnitude  $X_{21} = \text{Min}(6, 15) = 6$ . This exhausts the 1st column requirement. Hence, it is deleted from the table.

Finally, the allocation is made to the cell (2, 2) with magnitude  $X_{22} = \text{Min}(9, 9) = 9$  which satisfies the rim requirement. These allocation are shown in the transportation table as follows:

	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$O_1$	6	4	14	5	14
$O_2$	8	9	2	7	16
$O_3$	4	3	6	2	5
<b>Demand</b>	6	10	15	4	

(I Allocation)

	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$O_2$	8	9	1	7	15
$O_3$	4	3	6	2	5
<b>Demand</b>	6	10	4	4	

(II Allocation)

	$D_1$	$D_2$	$D_4$	Supply
$O_2$	8	9	7	15
$O_3$	4	3	2	1
<b>Demand</b>	6	10	0	

(III Allocation)

	$D_1$	$D_2$	Supply
$O_2$	8	9	15
$O_3$	4	3	0
<b>Demand</b>	6	9	

(IV Allocation)

	$D_1$	$D_2$	Supply
$O_2$	8	9	15
<b>Demand</b>	6	9	

(V, VI Allocation)

The following table gives the initial basic feasible solution.

	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$O_1$	6	4	1	5	14
			(14)		
$O_2$	8	9	2	7	16
	(6)	(9)	(1)		
$O_3$	4	3	6	2	5
		(1)		(4)	
<b>Demand</b>	6	10	15	4	

The solution is given by,

$$X_{13} = 14; X_{21} = 6; X_{22} = 9; X_{23} = 1; X_{32} = 1; X_{34} = 4$$

Transportation Cost,

$$\begin{aligned} &= 14 \times 1 + 6 \times 8 + 9 \times 9 + 1 \times 2 + 3 \times 1 + 4 \times 2 \\ &= 14 + 48 + 81 + 2 + 3 + 8 = ₹ 156 \end{aligned}$$

### 6.4.3 Vogel's Approximation Method (VAM)

The steps involved in this method for finding the initial solution are as follows.

**Step 1:** Find the penalty cost, namely the difference between the smallest and next smallest costs in each row and column.

**Step 2:** Among the penalties as found in Step 1 choose the maximum penalty. If this maximum penalty is more than one (i.e., if there is a tie) choose any one arbitrarily.

**Step 3:** In the selected row or column as by Step 2 find out the cell having the least cost. Allocate to this cell as much as possible depending on the capacity and requirements.

**Step 4:** Delete the row or column which is fully exhausted. Again, compute the column and row penalties for the reduced transportation table and then go to Step 2. Repeat the procedure until all the rim requirements are satisfied.

**Note:** If the column is exhausted, then there is a change in row penalty and vice versa.

**Example 6.7:** Find the initial basic feasible solution for the following transportation problem using VAM.

		Destination				Supply
		$D_1$	$D_2$	$D_3$	$D_4$	
Origin	$O_1$	11	13	17	14	250
	$O_2$	16	18	14	10	300
	$O_3$	21	24	13	10	400
	<b>Demand</b>	200	225	275	250	950

## NOTES

NOTES

**Solution:** Since  $\sum a_i = \sum b_j = 950$ , the problem is balanced and there exists a feasible solution to the problem.

First, we find the row and column penalty  $P_i$  as the difference between the least and the next least cost. The maximum penalty is 5. Choose the first column arbitrarily. In this column, choose the cell having the least cost name (1, 1). Allocate to this cell with minimum magnitude (i.e.,  $\text{Min}(250, 200) = 200$ .) This exhausts the first column. Delete this column. Since a column is deleted, then there is a change in row penalty  $P_{ii}$  and column penalty  $P_{ii}$  remains the same. Continuing in this manner, we get the remaining allocations as given in the following table below.

I Allocation

	$D_1$	$D_2$	$D_3$	$D_4$	Supply	$P_i$
$O_1$	11 200	13	17	14	50 250	2
$O_2$	16	18	14	10	300	4
$O_3$	21	24	13	10	400	3
<b>Demand</b>	200	225	275	250		
	0					
$P_i$	5 $\uparrow$	5	1	0		

II Allocation

	$D_2$	$D_3$	$D_4$	Supply	$P_{ii}$
$O_1$	13 50	17	14	50	1
$O_2$	18	14	10	300	4
$O_3$	24	13	10	400	3
<b>Demand</b>	225	275	250		
	175				
$P_{ii}$	5 $\uparrow$	1	0		

III Allocation

	$D_2$	$D_3$	$D_4$	Supply	$P_{iii}$
$O_2$	18 175	14	10	300 125	4
$O_3$	24	13	10	400	3
<b>Demand</b>	175	275	250		
	0				
$P_{iii}$	6 $\uparrow$	1	0		

IV Allocation

	$D_3$	$D_4$	Supply	$P_{iv}$
$O_2$	14	10 125	125 0	4 ←
$O_3$	13	10	400	3
<b>Demand</b>	275	250		
		125		
$P_{iv}$	1	0		

V Allocation

	$D_3$	$D_4$	Supply	$P_v$
$O_3$	13 275	10	400 125	3
<b>Demand</b>	275	125		
	0			
$P_v$	13 $\uparrow$	10		

VI Allocation

	$D_4$	Supply	$P_{vi}$
$O_3$	10 125	125 0	10 ←
<b>Demand</b>	125		
	0		
$P_{vi}$	10		

Finally, we arrive at the initial basic feasible solution which is shown in the following table.



	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$O_1$	11 (200)	13 (50)	17	14	250
$O_2$	16	18 (175)	14	10 (125)	300
$O_3$	21	24	13 (275)	10 (125)	400
<b>Demand</b>	200	225	275	250	

There are 6 positive independent allocations which equals to  $m + n - 1 = 3 + 4 - 1$ . This ensures that the solution is a non-degenerate basic feasible solution.

∴ Transportation Cost

$$= 11 \times 200 + 13 \times 50 + 18 \times 175 + 10 \times 125 + 13 \times 275 + 10 \times 125$$

$$= ₹ 12,075.$$

**Example 6.8:** Find the initial solution to the following TP using VAM.

		Destination				Supply
		$D_1$	$D_2$	$D_3$	$D_4$	
Factory	$F_1$	3	3	4	1	100
	$F_2$	4	2	4	2	125
	$F_3$	1	5	3	2	75
	<b>Demand</b>	120	80	75	25	300

**Solution:** Since  $\sum a_i = \sum b_j$  the problem is a balance TP. Hence, there exists a feasible solution.

	$D_1$	$D_2$	$D_3$	$D_4$	Supply	$P_I$	$P_{II}$	$P_{III}$	$P_{IV}$	$P_V$	$P_{VI}$
$F_1$	3 (45)	3	4 (30)	1 (25)	100	2	2	0	1	4	4
$F_2$	4	2 (80)	4 (45)	2	125	0	0	2	0	4	
$F_3$	1 (75)	5	3	2	75	1					
<b>Demand</b>	120	80	75	25							
$P_I$	2↑	1	1	1							
$P_{II}$	1	1	0	1							
$P_{III}$	1	1	0								
$P_{IV}$	1		0								
$P_V$			0								
$P_{VI}$			4↑								

Finally, we have the initial basic feasible solution as given in the following table.

## NOTES

**NOTES**

	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$F_1$	3 (45)	3	4 (30)	1 (25)	100
$F_2$	4	2 (80)	4 (45)	2	125
$F_3$	1 (75)	5	3	2	75
<b>Demand</b>	120	80	75	25	

There are 6 independent non-negative allocations equal to,

$$m + n - 1 = 3 + 4 - 1 = 6.$$

This ensures that the solution is non-degenerate basic feasible.

∴ Transportation Cost

$$\begin{aligned} &= 3 \times 45 + 4 \times 30 + 1 \times 25 + 2 \times 80 + 4 \times 45 + 1 \times 75 \\ &= 135 + 120 + 25 + 160 + 180 + 75 \\ &= ₹ 695. \end{aligned}$$

## 6.5 MODI METHOD

To perform this optimality test, we shall discuss the Modified Distribution method (MODI). The various steps involved in the MODI method for performing optimality test are as follows.

### MODI Method

**Step 1:** Find the initial basic feasible solution of a TP by using any one of the three methods.

**Step 2:** Find out a set of numbers  $u_i$  and  $v_j$  for each row and column satisfying  $u_i + v_j = C_{ij}$  for each occupied cell. To start with, we assign a number '0' to any row or column having maximum number of allocations. If this maximum number of allocations is more than one, choose any one arbitrarily.

**Step 3:** For each empty (unoccupied) cell, we find the sum  $u_i$  and  $v_j$  written in the bottom left corner of that cell.

**Step 4:** Find out for each empty cell the net evaluation value  $\Delta_{ij} = C_{ij} - (u_i + v_j)$  and which is written at the bottom right corner of that cell. This step gives the optimality conclusion,

- (i) If all  $\Delta_{ij} > 0$  (i.e., all the net evaluation value) the solution is optimum and a unique solution exists.
- (ii) If  $\Delta_{ij} \geq 0$  then the solution is optimum, but an alternate solution exists.
- (iii) If at least one  $\Delta_{ij} < 0$ , the solution is not optimum. In this case, we go to the next step, to improve the total transportation cost.

NOTES

**Step 5:** Select the empty cell having the most negative value of  $\Delta_{ij}$ . From this cell we draw a closed path by drawing horizontal and vertical lines with the corner cells occupied. Assign sign '+' and '-' alternately and find the minimum allocation from the cell having negative sign. This allocation should be added to the allocation having positive sign and subtracted from the allocation having negative sign.

**Step 6:** The previous step yields a better solution by making one (or more) occupied cell as empty and one empty cell as occupied. For this new set of basic feasible allocations, repeat from the Step 2 till an optimum basic feasible solution is obtained.

**Example 6.9:** Solve the following transportation problem.

		Destination				Supply
		P	Q	R	S	
Source	A	21	16	25	13	11
	B	17	18	14	23	13
	C	32	17	18	41	19
Demand		6	10	12	15	43

Origin/Dest	P	Q	R	S	Supply	$P_I$	$P_{II}$	$P_{III}$	$P_{IV}$	$P_V$	$P_{VI}$
A	21	16	25	13	11	3	-	-	-	-	-
B	17	18	14	23	13	3	4	4	4	-	-
C	32	17	18	41	19						
Demand	6	10	12	15	43						
$P_I$	4	1	4	10↑							
$P_{II}$	15	1	4	18↑							
$P_{III}$	15↑	1	4	-							
$P_{IV}$	-	1	4	-							
$P_V$	-	17	18↑	-							
$P_{VI}$	-	17↑	-	-							

**Solution:** We first find the initial basic feasible solution by using VAM. Since  $\sum a_i = \sum b_j$  the given TP is a balanced one. Therefore, there exists a feasible solution.

Finally, we have the initial basic feasible solution as given in the following table.

		Destination			
		P	Q	R	S
Source	A	21	16	23	13
	B	17	18	14	23
	C	32	17	18	41

From this table, we see that the number of non-negative independent allocations is  $6 = m + n - 1 = 3 + 4 - 1$ .

Hence, the solution is non-degenerate basic feasible.

∴ The Initial Transportation Cost

$$= 11 \times 13 + 3 \times 14 + 4 \times 23 + 6 \times 17 + 17 \times 10 + 18 \times 9 = ₹ 711.$$

**NOTES**

**To find the optimal solution:** We apply the MODI method in order to determine the optimum solution. We determine a set of numbers  $u_i$  and  $v_j$  for each row and column, with  $u_i + v_j = C_{ij}$  for each occupied cell. To start with we give  $u_2 = 0$  as the 2nd row has the maximum number of allocation.

Now, we find the sum  $u_i$  and  $v_j$  for each empty cell and enter at the bottom left corner of that cell.

$$C_{21} = u_2 + v_1 = 17 = 0 + v_1 = 17 \Rightarrow v_1 = 17$$

$$C_{23} = u_2 + v_3 = 14 = 0 + v_3 = 14 \Rightarrow v_3 = 14$$

$$C_{24} = u_2 + v_4 = 23 = 0 + v_4 = 23 \Rightarrow v_4 = 23$$

$$C_{14} = u_1 + v_4 = 13 = u_1 + 23 = 13 \Rightarrow u_1 = 10$$

$$C_{33} = u_3 + v_3 = 18 = u_3 + 14 = 18 \Rightarrow u_3 = 4$$

Next, we find the net evaluations  $\Delta_{ij} = C_{ij} - (u_i + v_j)$  for each unoccupied cell and enter at the bottom right corner of that cell.

**Initial table**

		$C_j$		4	3	0	0	-M	0	
$C_p$	B	$X_{ij}$	$W_j$	$W_j$	$S_1$	$S_2$	$A_1$	$S_1$		Min $\frac{X_{ij}}{W_j}$
0	$S_1$	1	1	1	1	0	0	0		1
-M	$A_1$	1	0	1	0	-1	1	0		1
$\leftarrow 0$	$S_2$	1	-1	②	0	0	0	1		$\frac{1}{2}$
	$Z_j$	-M	0	-M	0	M	-M	0		
	$Z_j - C_j$		-4	-M-3	0	M	0	0		Min $\frac{X_{ij}}{W_j}$
				↑						
$\leftarrow 0$	$S_1$	1/2	③	0	1	0	0	-1/2		1/3
-M	$A_1$	1/2	1/2	0	0	-1	1	-1/2		1
3	$W_2$	1/2	-1/2	1	0	0	0	1/2		-
	$Z_j$	$-\frac{1}{2}M-3/2$	$-\frac{1}{2}M-3/2$	3	0	M	-M	$\frac{1}{2}M+3/2$		
	$Z_j - C_j$		$-\frac{M}{2} + \frac{5}{2}$	0	0	M	0	$\frac{M+3}{2}$		
			↑							

Since all  $\Delta_{ij} > 0$ , the solution is optimal and unique. The optimum solution is given by,

$$X_{14} = 11; X_{21} = 6; X_{23} = 3; X_{24} = 4; X_{32} = 10; X_{33} = 9$$

The Minimum Transportation Cost,

$$= 11 \times 13 + 17 \times 6 + 3 \times 14 + 4 \times 23 + 10 \times 17 + 9 \times 18 = ₹ 711.$$

**Degeneracy in Transportation Problem**

In a TP, if the number of non-negative independent allocations is less than  $m + n - 1$ , where  $m$  is the number of origins (rows) and  $n$  is the number of destinations (columns) there exists a degeneracy. This may occur either at the initial stage or at the subsequent iteration.

To resolve this degeneracy, we adopt the following steps.

**Step 1:** Among the empty cell, we choose an empty cell having the least cost which is of an independent position. If this cell is more than one, choose any one arbitrarily.

**Step 2:** To the cell as chosen in Step 1 we allocate a small positive quantity  $\epsilon > 0$ .

The cells containing ' $\epsilon$ ' are treated like other occupied cells and degeneracy is removed by adding one (more) accordingly. For this modified solution, we adopt the steps involved in MODI method till an optimum solution is obtained.

**Example 6.10:** Solve the transportation problem for minimization.

		Destinations			Capacity
		1	2	3	
Sources	1	2	2	3	10
	2	4	1	2	15
	3	1	3	1	40
	Demand	20	15	30	65

**Solution:** Since  $\sum a_i = \sum b_j$  the problem is a balanced TP. Hence, there exists a feasible solution. We find the initial solution by North West Corner rule as given here.

	1	2	3	Capacity
1	(10) 2	2	3	10
2	(10) 4	(5) 1	2	15
3	1	(10) 3	(30) 1	40
Demand	20	15	30	

Since, the number of occupied cells = 5 =  $m + n - 1$ , and all the allocations are independent, we get an initial basic feasible solution.

The Initial Transportation Cost,

$$= 10 \times 2 + 4 \times 10 + 5 \times 1 + 10 \times 3 + 1 \times 30 = 20 + 40 + 5 + 30 + 30$$

$$= ₹ 125.$$

**To find the Optimal Solution (MODI Method):** We use the previous table to apply the MODI method. We find out a set of numbers  $u_i$  and  $v_j$  for which  $u_i + v_j = C_{ij}$ , only for occupied cell. To start with, as the maximum number of allocations is 2 in more than one row and column, we choose arbitrarily column 1 and assign a number 0 to this column, i.e.,  $v_1 = 0$ . The remaining numbers can be obtained as follows.

## NOTES

**NOTES**

$$C_{11} = u_1 + v_1 = 2 = u_1 + 0 = 2 \Rightarrow u_1 = 2$$

$$C_{21} = u_2 + v_1 = 4 \Rightarrow u_2 = 4 - 0 = 4$$

$$C_{22} = u_2 + v_2 = 1 \Rightarrow v_2 = 1 - u_2 = 1 - 4 = -3$$

$$C_{32} = u_3 + v_2 = 3 = u_3 = 3 - v_2 = 3 - (-3) = 6$$

$$C_{33} = u_3 + v_3 = 1 = v_3 = 1 - u_3 = 1 - 6 = -5$$

**Initial table**

	1	2	3	$u_i$
1	2 (10)	2 -1 3	3 -3 6	$2 = u_1$
2	4 (10) -	1 (5) +	2 -1 3	$4 = u_2$
3	1 6 +	3 (10) -	1 (30)	$6 = u_3$
$v_j$	$v_1 = 0$	$v_2 = -3$	$v_3 = -5$	

We find the sum of  $u_i$  and  $v_j$  for each empty cell and write at the bottom left corner of that cell. Find the net evaluations  $\Delta_{ij} = C_{ij} - (u_i + v_j)$  for each empty cell and enter at the bottom right corner of the cell.

The solution is not optimum as the cell (3, 1) has a negative  $\Delta_{ij}$  value. We improve the allocation by making this cell namely (3, 1) as an allocated cell. We draw a closed path from this cell and assign sign '+' and '-' alternately. From the cell having '-' sign we find the minimum allocation given by  $\text{Min}(10, 10) = 10$ . Hence, we get two occupied cells.

(2, 1) (3, 2) becomes empty and the cell (3, 1) is occupied and resulting in a degenerate solution. Degeneracy is discussed in subsequent iteration.

$$\text{Number of allocated cells} = 4 < m + n - 1 = 5.$$

We get degeneracy. To resolve we add the empty cell (1, 2) and allocate  $\epsilon > 0$ . This cell namely (1, 2) is added as it satisfies the two steps for resolving the degeneracy. We assign a number 0 to the first row, namely  $u_1 = 0$  we get the remaining numbers as follows.

$$C_{11} = u_1 + v_1 = 2 \Rightarrow v_1 = 2 - u_1 = 2 - 0 = 2$$

$$C_{12} = u_1 + v_2 = 2 \Rightarrow v_2 = 2 - u_1 = 2 - 0 = 2$$

$$C_{31} = u_3 + v_1 = 1 \Rightarrow u_3 = 1 - v_1 = 1 - 2 = -1$$

$$C_{33} = u_3 + v_3 = 1 \Rightarrow v_3 = 1 - u_3 = 1 - (-1) = 2$$

$$C_{22} = u_2 + v_2 = 1 \Rightarrow u_2 = 1 - v_2 = 1 - 2 = -1$$

Next, we find the sum of  $u_j$  and  $v_j$  for the empty cell and enter at the bottom left corner of that cell and also the net evaluation  $\Delta_{ij} = C_{ij} - (u_i + v_j)$  for each empty cell and enter at the bottom right corner of the cell.

**Iteration table**

	1	2	3	$u_i$
1	⑩	⊕	2	0
2	1	⑮	1	-1
3	⑩	1	③①	-1
$v_j$	2	2	2	

**NOTES**

The modified solution is given in the following table. This solution is also optimal and unique as it satisfies the optimality condition that all  $\Delta_{ij} > 0$ .

	1	2	3	Capacity
1	⑩	⊕	3	10
2	4	⑮	2	15
3	1	3	1	40
<b>Demand</b>	②①	15	③①	65

$$X_{11} = 10; X_{22} = 15; X_{33} = 30; X_{12} = \epsilon; X_{31} = 10$$

$$\text{Total Cost} = 10 \times 2 + 2 \times \epsilon + 15 \times 1 + 10 \times 1 + 30 \times 1$$

$$= 75 + 2 \epsilon = ₹ 75.$$

**Unbalanced Transportation**

The given TP is said to be unbalanced if  $\sum a_i \neq \sum b_j$ , i.e., if the total supply is not equal to the total demand.

There are two possible cases.

**Case (i):**  $\sum_{i=1}^m a_i < \sum_{j=1}^n b_j$

If the total supply is less than the total demand, a dummy source (row) is included in the cost matrix with zero cost; the excess demand is entered as a rim requirement for this dummy source (origin). Hence, the unbalanced transportation problem can be converted into a balanced TP.

**Case (ii):**  $\sum_{i=1}^m a_i > \sum_{j=1}^n b_j$

**NOTES**

Therefore, the total supply is greater than the total demand. In this case, the unbalanced TP can be converted into a balanced TP by adding a dummy destination (column) with zero cost. The excess supply is entered as a rim requirement for the dummy destination.

**Example 6.11:** Solve the transportation problem when the unit transportation costs, demands and supplies are as given below:

		Destination				Supply
		$D_1$	$D_2$	$D_3$	$D_4$	
Origins	$O_1$	6	1	9	3	70
	$O_2$	11	5	2	8	55
	$O_3$	10	12	4	7	70
	Demand	85	35	50	45	

**Solution:** Since the total demand  $\sum b_j = 215$  is greater than the total supply  $\sum a_i = 195$ , the problem is an unbalanced TP.

We convert this into a balanced TP by introducing a dummy origin  $O_4$  with cost zero and giving supply equal to  $215 - 195 = 20$  units. Hence, we have the converted problem as follows:

		Destination				Supply
		$D_1$	$D_2$	$D_3$	$D_4$	
Origins	$O_1$	6	1	9	3	70
	$O_2$	11	5	2	8	55
	$O_3$	10	12	4	7	70
	$O_4$	0	0	0	0	20
	Demand	85	35	50	45	215

As this problem is balanced, there exists a feasible solution to this problem. Using VAM we get the initial solution.



	$D_1$	$D_2$	$D_3$	$D_4$	Supply	$P_1$	$P_{II}$	$P_{III}$	$P_{IV}$	$P_V$	$P_{VI}$	$P_{VII}$
$O_1$	6 (65)	1 (5)	9	3	70	2	2	2	-	-	-	-
$O_2$	11	5 (30)	2 (25)	8	55	3	3	3	3	6 ←	-	-
$O_3$	10	12	4 (25)	7 (45)	70	3	3	3	3	3	3	4 ←
$O_4$	0 (20)	0	0	0	20	0	-	-	-	-	-	-
<b>Demand</b>	85	35	50	45								
$P_1$	6↑	1	2	3								
$P_{II}$	4↑	4	2	3								
$P_{III}$	-	4↑	2	4								
$P_{IV}$	-	7↑	2	1								
$P_V$	-	-	2	1								
$P_{VI}$	-	-	4	7↑								
$P_{VII}$	-	-	4	-								

The initial solution to the problem is given by,

	$D_1$	$D_2$	$D_3$	$D_4$
$O_1$	6 (65)	1 (5)	9	3
$O_2$	11	5 (30)	2 (25)	8
$O_3$	10	12	4 (25)	7 (45)
$O_4$	0 (20)	0	0	0

There are 7 independent non-negative allocations equal to  $m + n - 1$ . Hence, the solution is a non-degenerate one.

The Total Transportation Cost

$$= 6 \times 65 + 5 \times 1 + 5 \times 30 + 2 \times 25 + 4 \times 25 + 7 \times 45 + 20 \times 0$$

$$= ₹ 1010.$$

**To find the Optimal Solution:** We apply the steps in the MODI method to the previous table.

## NOTES

**NOTES**

	$D_1$	$D_2$	$D_3$	$D_4$	$u_i$
$O_1$	(65) - 6	+ (5) 1	0 9	3 3	0
$O_2$	10 11	(30) 5	+ (25) 2	5 8	4
$O_3$	12 10	-2 7 5	(25) 4	(45) 7	6
$O_4$	(20) 0	-5 5	-4 4	-5 5	
$v_j$	6	1	-2	1	

Since all  $\Delta_{ij} \geq 0$  the solution is not optimum and to solve we introduce the cell (3,1) as this cell has the most negative value of  $\Delta_{ij}$ . We modify the solution by adding and subtracting the minimum allocation given by Min (65, 30, 25). While doing this, the occupied cell (3, 3) becomes empty.

**First iteration table**

	$D_1$	$D_2$	$D_3$	$D_4$	$u_i$
$O_1$	(40) 6	(30) 1	-2 9	3 3	6
$O_2$	10 11	(5) 5	(50) 2	7 8	10
$O_3$	(25) 10	5 12	2 4	(45) 7	10
$O_4$	(20) 0	-5 5	-8 8	-3 3	0
$v_j$	0	-5	-8	-3	

As the number of independent allocations are equal to  $m + n - 1$ , we check the optimality.

Since, all  $\Delta_{ij} \geq 0$ , the solution is optimal and an alternate solution exists as  $\Delta_{14} = 0$ . Therefore, the optimum allocation is given by,

$$X_{11} = 40; X_{12} = 30; X_{22} = 5; X_{23} = 50; X_{31} = 25; X_{34} = 45; X_{41} = 20$$

The Optimum Transportation Cost,

$$= 6 \times 40 + 1 \times 30 + 5 \times 5 + 2 \times 50 + 10 \times 25 + 7 \times 45 + 0 \times 20$$

$$= ₹ 960.$$

**Check Your Progress**

1. What is a transportation problem?
2. What must one know to achieve the objectives of a transportation problem?
3. Give reasons why the LPP solution technique is not used for solving a transportation problem.
4. What is the purpose of the MODI method?

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## 6.6 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

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1. Transportation problem deals with transportation of various quantities of a single homogeneous commodity initially stored at various origins to different destinations at the minimum cost.
2. To achieve the objectives, one must know: (i) Amount and location of available supplies, (ii) Quantity demanded at destination and (iii) Costs of transporting one unit of commodity from various origins to various destinations.
3. As there are  $m + n - 1$  equations in a transportation problem with  $m$  origins and  $n$  destinations, by adding an artificial variable to each equation, a large number of variables are involved.
  - (i) If the problem has  $m$  sources and  $n$  destinations and  $m + n - 1$  equations can be formed. Hence, computation may exceed the capacity of the computer. So LPP solution is not made use for solving a TP.
  - (ii) The coefficient  $X_{ij}$  in the constraints are all in unity. For such a technique, transportation technique is easier than simplex method.
  - (iii) TP is minimization of objective function, whereas, simplex method is suitable for maximization problem.
4. The purpose of MODI method is to get the optimal solution of a transportation problem.

## NOTES

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## 6.7 SUMMARY

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- Transportation problem deals with transportation of various quantities of a single homogeneous commodity initially stored at various origins to different destinations and at the minimum cost. This is a subclass of a Linear Programming Problem (LPP).
- The initial solution of a transportation problem is found using any of three rules, North West Corner Rule, Least Cost Method or Vogel's Approximation Method (VAM). The last one is preferred since the solution obtained this way is very close to the optimal solution.
- The MODI method is used to obtain optimal solutions and to carry out optimality tests.
- Any set of non-negative allocations ( $X_{ij} > 0$ ) which satisfies the row and column sum (rim requirement) is called a feasible solution.
- A feasible solution is called a basic feasible solution if the number of non-negative allocations is equal to  $m + n - 1$ , where  $m$  is the number of rows and  $n$  the number of columns in a transportation table.
- Any feasible solution to a transportation problem containing  $m$  origins and  $n$  destinations is said to be non-degenerate, if it contains  $m + n - 1$  occupied cells and each allocation is in independent positions.

## NOTES

- The objective of travelling salesman is to select the sequence in which the cities are visited in such a way that his total travelling time is minimized.
- The cells in the transportation table can be classified as occupied cells and unoccupied cells. The allocated cells in the transportation table are called occupied cells and empty cells in a transportation table are called unoccupied cells.
- Optimality test can be conducted to any initial basic feasible solution of a TP provided such allocations has exactly  $m + n - 1$  non-negative allocations, where  $m$  is the number of origins and  $n$  is the number of destinations. Also, these allocations must be in independent positions.
- In a TP, if the number of non-negative independent allocations is less than  $m + n - 1$ , where  $m$  is the number of origins (rows) and  $n$  is the number of destinations (columns) there exists a degeneracy. This may occur either at the initial stage or at the subsequent iteration.
- The given TP is said to be unbalanced if  $\sum a_i \neq \sum b_j$ , i.e., if the total supply is not equal to the total demand.
- In a transshipment model, the objects are supplied from various specific sources to various specific destinations. It is also economic if the shipment passes via the transient nodes which are in between the sources and the destinations.

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## 6.8 KEY WORDS

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- **Feasible solution:** Any set of non-negative allocations where some quantity is transferred from an origin  $i$  to a destination  $j$ , ( $X_{ij} > 0$ ) and satisfies the row and column sum is a feasible solution
- **Non-degenerate basic feasible solution:** Any feasible solution to a transportation problem containing  $m$  origins and  $n$  destinations is known as non-degenerate, if it contains  $m + n - 1$  occupied cells and each allocation is in independent positions
- **Degenerate basic feasible solution:** A basic feasible solution that contains less than  $m + n - 1$  non-negative allocations

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## 6.9 SELF ASSESSMENT QUESTIONS AND EXERCISES

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### Short Answer Questions

1. What is an unbalanced transportation problem?
2. How will you convert an unbalanced transportation problem into a balanced one?

3. List the merits and limitations of the North West corner rule.
4. Vogel's approximation method results in the most economical initial basic feasible solution. How?
5. What is the number of non-basic variables in a balanced transportation problem?
6. While dealing with North West corner rule, when does one move to the next cell in next column?
7. What kind of solution would you get when net change in value of all unoccupied cells is non-negative?

## NOTES

### Long Answer Questions

1. Define feasible solution, basic solution, non-degenerate solution, optimal solution in a transportation problem.
2. Explain the following briefly with examples:
  - (i) North West Corner Rule
  - (ii) Least Cost Method
  - (iii) Vogel's Approximation Method
3. Explain degeneracy in a transportation problem. Describe a method to resolve it.
4. Obtain the initial solution for the following transportation problem using (i) North West Corner Rule (ii) Least Cost Method (iii) VAM

		<i>Destination</i>			<i>Supply</i>
		<i>A</i>	<i>B</i>	<i>C</i>	
<i>Source</i>	1	2	7	4	5
	2	3	3	1	8
	3	5	4	7	7
	4	1	6	2	14
	<i>Demand</i>	7	9	18	34

5. Solve the following transportation problem using Vogel's approximation method.

		<i>Warehouse</i>						<i>Available</i>
		<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	
<i>Factory</i>	1	9	12	9	6	9	10	5
	2	7	3	7	7	5	5	6
	3	6	5	9	11	3	11	2
	4	6	8	11	2	2	10	9
	<i>Requirement</i>	4	4	6	2	4	2	

**NOTES**

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**6.10 FURTHER READINGS**

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# UNIT 7 THEORY OF PROBABILITY

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## Structure

- 7.0 Introduction
- 7.1 Objectives
- 7.2 Introduction to the Concept: Development
  - 7.2.1 The Concept of Sample Space, Sample Points and Events
- 7.3 Types of Probability: Utilisation in Business
- 7.4 Permutations and Combinations
  - 7.4.1 Permutations
  - 7.4.2 Combinations
- 7.5 Answers to Check Your Progress Questions
- 7.6 Summary
- 7.7 Key Words
- 7.8 Self Assessment Questions and Exercises
- 7.9 Further Reading

## NOTES

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## 7.0 INTRODUCTION

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In this unit, you will learn different theories of probability and will understand why probability is considered the most important tool in statistical calculations. The subject of probability in itself is a cumbersome one, hence only the basic concepts will be discussed in this unit. The word probability or chance is very commonly used in day-to-day conversation, and terms such as possible or probable or likely, all have the similar meaning. Probability can be defined as ‘a measure of the likelihood that a particular event’ will occur. It is a numerical measure with a value between 0 and 1 of such likelihood where the probability of zero indicates that the given event cannot occur and the probability of one assures certainty of such an occurrence. The probability theory helps a decision-maker to analyse a situation and decide accordingly. The important types of probability, viz., a-priori and empirical probability, objective and subjective probability are explained with the help of solved examples.

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## 7.1 OBJECTIVES

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After going through this unit, you will be able to:

- Understand the basic concept of probability
- Explain sample space, sample points and events
- Describe the important types of probability
- Explain permutations and combinations

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## 7.2 INTRODUCTION TO THE CONCEPT: DEVELOPMENT

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### NOTES

Probability (usually represented by the symbol ‘P’) may be defined as the percentage of times in which a specific outcome would happen if an event was repeated a very large number of times. In other words, the probability of the occurrence of an event is the ratio of the number of times the **event** occurs (or can occur) to the number of times it and all other events occur (or can occur).

The general meaning of the word ‘Probability’ is likelihood. Where the happening of an event is certain, the probability is said to be unity, i.e., equal to 1 and where there is absolute impossibility of happening of an event, the probability is said to be zero. But in real life such cases are rare and the probability generally lies between 0 and 1. Thus, probabilities are always greater than or equal to zero (i.e., probabilities are never negative) and are equal to or less than one. This being so, we can say that the weight scale of probability runs from zero to one and in symbolic form it can be stated as follows, i.e.,

$$P \leq 1 \text{ but } \geq 0$$

Probability can be expressed either in terms of a fraction or a decimal or a percentage but generally it is expressed in decimals.

### 7.2.1 The Concept of Sample Space, Sample Points and Events

A **sample space** refers to the complete set of outcomes for the situation as it did or may exist. An element in a set serving as a sample space is called a *sample point*. An *event* is a statement which refers to a particular subset of a sample space for an experiment. The meaning of these three concepts can be easily understood by means of an example. Let us consider an experiment of tossing first one coin and then another. The sample space relevant to it would then consist of all the outcomes of this experiment and can be stated as under:

$$S = [HH, HT, TH, TT]$$

It may be noted that  $S = [ \quad ]$  is the symbol used to represent the sample space. This sample space has four outcomes or what we call sample points viz.,  $HH, HT, TH$ , and  $TT$ . One or more of these sample points are called an event. One event may be that both coins fall alike and this can be represented as,

$$E_1 = [HH, TT; \text{alike}]$$

The word following the semicolon explains the characteristic of our interest. If  $E_1$  be the event of our interest and  $E_2$  be the subset of all the remaining outcomes then we have the following equation:

$$S = E_1 + E_2$$



### 7.3 TYPES OF PROBABILITY: UTILISATION IN BUSINESS

Some important types of probabilities are given as follows:

(1) A-priori probability and Empirical probability; (2) Objective probability and Subjective probability; (3) Marginal, Conditional and Joint probabilities.

Here we are explaining only the first two probabilities.

#### (1) A-priori Probability and Empirical Probability

As long as any outcome or sample point concerning an experiment is not affected by external factors, each outcome or sample point is equally likely to occur. This assumption of each sample point being equally likely to occur is known as an a-priori assumption (the term a-priori refers to something which is known by reason alone) and the probability of an event worked out on this assumption is known as 'a-priori probability'. Thus, a-priori probability is one which is worked out through deduction from assumed principles. A-priori probability is also termed as classical probability. In our example, each sample point occurs with equal frequency.  $E_1$  occurs twice and  $E_2$  occurs twice, thus the probability of  $E_1$  is:

$$P(E_1) = \frac{E_1}{E_1 + E_2} = \frac{1}{2}$$

and the probability of  $E_2$  is:

$$P(E_2) = \frac{E_2}{E_1 + E_2} = \frac{1}{2}$$

These are the examples of a-priori probabilities. One does not need to toss the two coins a large number of times in order to predict such probabilities. A-priori probabilities are also known as mathematical probabilities and are associated with games of chance including throws of a coin or a dice. If a coin is thrown, the probability that it will be head upwards is  $\frac{1}{2}$  (half) since we know that the number of possible alternatives in this case are two only. Similarly, the probability of getting a five in the single throw of a dice is  $\frac{1}{6}$  (one sixth), since we know that possible alternatives in this case are six only.

*Empirical probability* (or statistical probability) is based on recording actual experience over a period of time and computing the proportion of items that each event occurred. Empirical probability of an event may be expressed as:

$$P(E) = \frac{\text{Total number of occurrences of the event } E}{\text{Total number of trials}}$$

For example, if the coin has been thrown 200 times and the head coming up was noticed in this experiment 120 times, the empirical probability of head coming up would be equal to  $\frac{120}{200}$  or 0.6. 'The empirical probability of an event is taken

#### NOTES

**NOTES**

as the relative frequency of occurrence of the event when the number of observations is very large.' We can also state that if on taking a very large number  $N$  out of a series of cases in which an event  $E$  is in question,  $E$  happens on  $pN$  occasions, the probability of the event  $E$  is said to be  $p$ . According to the laws of inertia of large numbers and statistical regularity, the more the trials, the more the chance that empirical probability would move towards nearer and nearer and finally may become equal to a-priori or the mathematical probability.

**(2) Objective and Subjective Probabilities**

Objective probabilities are those which are based on definite historical information, common experience (objective evidence) or some rigorous analysis but in case of subjective probabilities it is the personal experience alone which becomes the basis of the probability assignment. Let us illustrate this by an example. Suppose we have a box which contains 5 black and 15 white balls. If the balls are mixed thoroughly, then we would assign an objective probability of  $\frac{5}{20}$  of drawing a black ball and  $\frac{15}{20}$  of drawing a white ball. Similarly, the probabilities of various events in throwing of dice or in tossing coins are examples of objective probabilities, since they may be and generally are based on reliable objective evidences. But imagine a situation wherein a businessman is trying to decide whether or not to buy a new factory and the success of the factory largely depends on whether there is a recession or not in the next four years. If a probability is assigned to the occurrence of a recession, it would be a subjective weight based on the personal experience of the businessman. Such probability constitutes an example of subjective probability. For business decision-making purposes the subjective probabilities are frequently required and used, specially because of the fact that reliable objective evidences are not always available.

**(3) Joint, Conditional and Marginal Probabilities**

Joint probability is the type of probability when two events occur simultaneously. Marginal probability is the probability of the occurrence of the single event. The conditional probability is the probability that event  $A$  will occur given that or on the condition that event  $B$  has already occurred. It is denoted by  $P(A|B)$ . Joint probability is not the same as conditional probability. Conditional probability assumes that one event has taken place or will take place and then asks for the probability of the other ( $A$ , given  $B$ ). Joint probability does not have such conditions, it simply asks for the chances of both happening ( $A$  and  $B$ ).

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**7.4 PERMUTATIONS AND COMBINATIONS**


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For every student of statistics and quantitative techniques, a knowledge of Permutations and Combinations forming part of algebra is absolutely essential. An elementary of these two concepts is presented here.

### 7.4.1 Permutations

**Permutations** refer to the different ways in which a number of objects can be arranged in a definite order. For instance, suppose there are two things  $x$  and  $y$ . They can be arranged in two different ways, i.e.,  $xy$  and  $yx$ . These two arrangements are called permutations. Similarly if there are three things  $x$ ,  $y$  and  $z$  then they all can be arranged in the following ways:

$$xyz; xzy; yxz; yzx; zxy; zyx$$

These six arrangements are the concerning permutations in the given case. But if we want to have only any two things out of three things in our arrangements then the following arrangements would be possible:

$$xy . xz . zy$$

$$yx . zx . yz$$

These six arrangements are the concerning permutations in this case.

*The word permutation thus refers to the arrangements which can be made by taking some or all of a number of things.*

#### Formulae Concerning Permutations

(1) *Formula concerning permutations of  $n$  different things taken  $r$  at a time:*

$$\text{Number of Permutations} = {}^n P_r = \frac{n!}{(n-r)!}$$

Where,  $n$  = Number of different things given.

$r$  = Number of different things taken at a time out of different things given.

$n!$  is read as 'n factorial' and it means the product of 1, 2, 3, 4, ...n.

**Note:**  $0!$  is read as 'zero factorial' and is always taken as equal to 1.

**Example 7.1:** How many four letter words can be made using the letters of the word JAIPUR?

**Solution:**

Since,  $n = 6$

and  $r = 4$

$$\begin{aligned} \text{Number of Permutations, } {}^n P_r &= \frac{n!}{(n-r)!} = \frac{6!}{(6-4)!} = \frac{6!}{2!} \\ &= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 360 \end{aligned}$$

Thus the four letter words can be made in 360 ways using the letters of the word JAIPUR.

## NOTES

**Example 7.2:** How many permutations are possible of the letters of the word PROBABILITY when taken all at a time?

**Solution:**

### NOTES

Since,  $n = 11$

and  $p = 2$  (as letter  $B$  is occurring twice in the given word)

$q = 2$  (as letter  $I$  is occurring twice in the given word)

and all other letters in the given word are different.

The required number of permutations is

$$\begin{aligned} &= \frac{11!}{2!2!} \\ &= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = 9979200 \end{aligned}$$

### 7.4.2 Combinations

**Combinations** refer to the number of arrangements which can be made from a group of objects irrespective of their order. Combinations differ from permutations in respect that one combination, for example,  $xyz$  may be stated in the form of several permutations just by rearranging the order such as:

$$xyz; xzy; yxz; yzx; zxy; zyx$$

All of these are one combination but they are six permutations. The number of permutations is always greater than the number of combinations in any given situation, since a combination of  $n$  different things can generate  $n!$  permutations.

The number of combinations of  $n$  different things taken  $r$  at a time is given by the following formula:

$$\text{Number of Combinations} = {}^n C_r = \frac{n!}{(n-r)!r!}$$

Where,  $n$  = Number of different things given.

$r$  = Number of different things taken at a time out of different things given.

**Example 7.3:** In how many ways can four persons be chosen out of seven?

**Solution:**

Since,  $n = 7$

and  $r = 4$

$$\text{Number of combinations } {}^n C_r = {}^7 C_4 = \frac{7!}{(7-4)!4!} = \frac{7!}{3!4!}$$

$$= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1} = 35$$

Thus four persons out of seven can be chosen in 35 ways.

### Some Other Rules

(a) If an operation can be performed in  $x$  ways and having been performed in one of these ways, a second operation can then be performed in  $y$  ways, the number of performing the two operations would be  $x.y$ . This rule can be extended to cases where there are more than two operations to be performed.

**Example 7.4:** There are five different routes from Jaipur to Agra. In how many ways can a person go from Jaipur to Agra by one route and return by another route?

**Solution:** The man can go from Jaipur to Agra in five ways but while returning he can come to Jaipur in four ways only as he cannot adopt the route through which he went to Agra. The total number of ways of performing the journey in the prescribed manner in thus  $5 \times 4 = 20$ .

(b) The number of ways in which  $x + y + z$  things can be divided into three groups

containing  $x$ ,  $y$  and  $z$  things respectively is,

$$= \frac{(x + y + z)!}{x!y!z!}$$

**Example 7.5:** In how many ways can 10 books be put to three shelves which can contain 2, 3 and 5 books respectively?

**Solution:** There are three shelves in which ten books are to be put. This means the groups are to be of 2, 3 and 5 books. This can be done as follows,

$$\frac{(2+3+5)!}{2!3!5!} = \frac{10!}{2!3!5!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 3 \times 2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1} = 2520$$

The 10 books can be put to three shelves in 2520 ways as per the above rule.

### Check Your Progress

1. Define probability.
2. In what situations does one need probability theory?
3. What do you mean by sample space, sample points and events?
4. What are the various types of probabilities?
5. What is the difference between a-priori and empirical probabilities?

### NOTES

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## 7.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

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### NOTES

1. Probability can be defined as the percentage of times in which a specific outcome would happen if an event were repeated a very large number of times. In other words, probability of the occurrence of an event is the ratio of the number of times the event can occur to the number of times it and all other events can occur. Probability is usually represented by the symbol,  $P$ .
2. Probability theory is needed in situations when one involved in business or industrial field needs to make predictions.
3. A sample space is the complete set of outcomes for the situation as it did or may exist, while an element in a set serving as a sample space is called a sample point. An event is a statement that refers to a particular subset of a sample space for an experiment.
4. The various types of probabilities are:
  - (a) A-priori probability and empirical probability
  - (b) Objective probability and subjective probability
  - (c) Marginal, conditional and joint probabilities
5. As long as any outcome or sample point concerning an experiment is not affected by external factors, each outcome or sample point is equally likely to occur. This assumption of each sample point being equally likely to occur is known as an a-priori assumption and the probability of an event worked out on this assumption is known as a-priori probability. On the other hand, empirical probability is based on recording actual experience over a period of time and computing the proportion of items that each event occurred.

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## 7.6 SUMMARY

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- In decision-making processes, mathematical theory of probability is considered as an important and fundamental tool and is used by the decision-makers to evaluate the outcome of an experiment.
- Probability is a measure of how likely an event is to happen or in other terms can be used to estimate frequencies of outcomes in random experiments, which is always greater than or equal to zero.
- Permutations and combinations are discussed and analysed to calculate the probability.
- Various business decisions in real life are made under situations when the decision-maker is uncertain as to how their decisions are likely to result.
- Mathematical theory of probability furnishes an important tool which can be of great help to the decision-maker.

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## 7.7 KEY WORDS

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- **Event:** It is an outcome or a set of outcomes of an activity or a result of a trial
- **Sample space:** It is the collection of all possible events or outcomes of an experiment
- **Empirical probability:** Empirical probability (or statistical probability) is based on recording actual experience over a period of time and computing the proportion of items that each event occurred

## NOTES

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## 7.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

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### Short Answer Questions

1. State the concept of probability.
2. What are the important theories of probability? State briefly.
3. Differentiate between objective and subjective probabilities.
4. Differentiate between permutations and combinations with the help of examples.

### Long Answer Questions

1. What do you understand by the term probability? State and prove the addition theorem of probability.
2. Distinguish between (a) A-priori and Empirical probabilities and (b) Objective and Subjective probabilities.
3. Define the concepts of sample space, sample points and events in context of probability theory.

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## 7.9 FURTHER READINGS

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- Taha, H.A. 2006. *Operations Research: An Introduction*, 8th edition. New Delhi: Prentice-Hall of India.
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NOTES

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# UNIT 8 THEORETICAL PROBABILITY DISTRIBUTIONS

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## Structure

- 8.0 Introduction
- 8.1 Objectives
- 8.2 Concept and Probability of Events
  - 8.2.1 In Case of Simple Events
  - 8.2.2 In Case of Mutually Exclusive Events
  - 8.2.3 In Case of Compound Events
- 8.3 Types of Probability Distributions
  - 8.3.1 Binomial Distribution
  - 8.3.2 Poisson Distribution
  - 8.3.3 Normal Distribution
- 8.4 Answers to Check Your Progress Questions
- 8.5 Summary
- 8.6 Key Words
- 8.7 Self Assessment Questions and Exercises
- 8.8 Further Readings

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## 8.0 INTRODUCTION

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The binomial distribution is used in finite sampling problems where each observation is one of two possible outcomes ('success' or 'failure'). The Poisson distribution is used for modeling rates of occurrence. The exponential distribution is used to describe units that have a constant failure rate. The term 'normal distribution' refers to a particular way in which observations will tend to pile up around a particular value rather than be spread evenly across a range of values.

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## 8.1 OBJECTIVES

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After going through this unit, you will be able to:

- Discuss the concept and probability of events
- Explain the types of probability distribution
- Describe the binomial distribution based on Bernoulli process
- Describe the Poisson distribution
- Analyse Poisson distribution as an approximation of binomial distribution
- Explain the basic theory, characteristics and family of normal distributions



## 8.2 CONCEPT AND PROBABILITY OF EVENTS

The following are the methods of calculation of probability.

### 8.2.1 In Case of Simple Events

**Simple events** are also known as single events. In such events the probability principle involved is that if an event can happen in  $a$  ways and fail to happen in  $b$  ways, all these ways being equally likely and such that not more than one of them can occur, then the probability of the event happening is  $\frac{a}{(a+b)}$  and the probability

of its not happening is  $\frac{b}{(a+b)}$ . The probability of happening of an event and the probability of its not happening must always sum to one. If the probability of happening of an event being given as  $\frac{a}{(a+b)}$  then the possibility of its not happening

can be easily worked out as  $1 - \frac{a}{(a+b)}$  and it would be equal to  $\frac{b}{(a+b)}$ .

In simplest problems on probability the number of favourable ways and the total number of ways in which the event can happen, can be counted either arithmetically or by the help of simple rules (permutations and combinations have been discussed in the 5.6 of this unit) and after that the probability of happening can simply be worked out by dividing the number of favourable ways by the number of total ways in which an event can happen.

Thus,

Probability of happening of an event,

$$= \frac{\text{Number of favourable ways}}{\text{Total number of ways in which an event can happen}}$$

**Example 8.1:** Find the probability that if a card is drawn at random from an ordinary pack, it is one of the court cards.

**Solution:** Since there are 52 cards in an ordinary pack, the total number of ways in which a card can be drawn = 52.

The number of favourable ways = 12 because King, Queen and Jack of each of the four colours is to be included in court cards.

$$\text{Hence, the required probability is } = \frac{12}{52} = \frac{3}{13}$$

**Note:** Sometimes the information given may be in the form of odds in favour or odds against. If the odds are in favour of the event as  $a$  to  $b$  (or odds against  $b$  to  $a$ ) then the probability

of the happening of the event is  $= \frac{a}{(a+b)}$ .

### NOTES

## NOTES

**Example 8.2:** Odds in favour of  $A$  solving the problem are 6 : 8. Find the probability of  $A$  solving the problem.

**Solution:** The given information means that out of 14 times,  $A$  can solve the problem 6 times and fails to solve the problem of 8 times. Hence, the probability of solving the problem is,

$$= \frac{6}{14} = \frac{3}{7}$$

### 8.2.2 In Case of Mutually Exclusive Events

If two events do not happen on any one occasion, then the events are known as mutually exclusive events. In other words, events are said to be mutually exclusive when only one of the events can occur on any one trial. The probabilities of these events can be added to obtain the probability that at least one of a given collection of the events will occur. This is known as the **additional rule** of probability. It can be stated as, if an event can happen in more than one way, all ways being mutually exclusive, the probability of its happening at all is the sum of the probabilities of its happenings in the several ways. In terms of set theory we can state this probability relationship as under:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

Provided the events  $A$ ,  $B$ , and  $C$  are mutually exclusive, i.e.,  $A$ ,  $B$  and  $C$  do not intersect at any point. But if they intersect (in that case events  $A$ ,  $B$ , and  $C$  are not mutually exclusive) then the probability relationship will have to be modified and can be stated as under:

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &= -P(A \cap B) - P(A \cap C) \\ &= -P(B \cap C) + P(A \cap B \cap C) \end{aligned}$$

**Example 8.3:** Find the chance of throwing a number greater than 4 with an ordinary dice.

**Solution:** An ordinary dice has six faces marked as 1 to 6. Numbers greater than four can be 5 or 6. These numbers are mutually exclusive which means if we throw 5 we can not throw 6 simultaneously and if we throw 6 we can not throw 5 at the same time. The probability of throwing five with a single dice is  $1/6$  and the probability of throwing six with a single dice is also  $1/6$ . Hence, the required probability of throwing either 5 or 6 (i.e., a number greater than four) with a single dice is,

$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

**Example 8.4:** From a set of 17 cards numbered as 1, 2 . . . . 17, one is drawn at random. What is the probability that the card drawn bears a number which is divisible by 3 or 7?

**Solution:** In this example the numbers divisible by three are 3, 6, 9, 12, 15 and the numbers divisible by seven are 7 and 14 only. The card drawn at random may bear any of these numbers (which are mutually exclusive) and the probability of each of these numbers coming is  $1/17$ . Hence, the probability that the card drawn bears a number which is divisible by 3 or 7 is,

$$= \frac{1}{17} + \frac{1}{17} + \frac{1}{17} + \frac{1}{17} + \frac{1}{17} + \frac{1}{17} + \frac{1}{17} = \frac{7}{17}$$

## NOTES

### 8.2.3 In Case of Compound Events

When two or more events occur together, their joint occurrence is called a **compound event**. The two more events occurring together and constituting a compound event may be either independent or dependent. If two events are independent (statistically), the occurrence of the one event will not affect the probability of the occurrence of the second event. On the other hand two events are said to be dependent if the occurrence of one of the events affects the probability of the occurrence of the second event. For instance, if a bag contains 10 balls and one ball is drawn from it and it is not replaced back and then a second ball is drawn, the drawing of the second ball is dependent on that of the first. But if the ball drawn is replaced, the second drawing of the ball will be taken as independent of the first. Another example of dependent events involves mutually exclusive events. If events  $A$  and  $B$  are mutually exclusive, they are dependent. Given that event  $A$  has occurred then probability of event  $B$  occurring must be zero since the two cannot happen simultaneously.

*The probability principle in case of independent events.* When two (or more) events are independent the probability of both events (or more than two events) occurring together or in succession is equal to the product of the chances of their happening separately. This can be stated as:

$$P(A \cap B) \text{ or } P(AB) = P(A).P(B)$$

This equation shows that the probability of events  $A$  and  $B$  both occurring is equal to the probability of events  $A$  times the probability of event  $B$ , if  $A$  and  $B$  are independent events. The probability of both events  $A$  and  $B$  occurring together, i.e.,  $P(AB)$  or  $P(A \cap B)$  is also known as *joint probability* of events  $A$  and  $B$ . *The rule concerning probability applicable in such a case is known as the multiplication theorem of probability.*

To define independence statistically we sometimes need the symbol:

$$P(B/A)$$

This symbol is read as ‘the probability of event  $B$  given that event  $A$  has occurred.’ This also indicates the *conditional probability* of event  $B$  given that event  $A$  has taken place. With independent events:

$$P(B/A) = P(B)$$

and similarly,

$$P(A/B) = P(A)$$

Thus, *with two independent events, the occurrence of one event does not affect the probability of the occurrence of the second event.*

*The probability principle in case of dependent events.* The probability principle in case of dependent events remains the same as in the case of independent events. If  $P(A)$  is the probability of happening of an event  $A$  and  $P(B/A)$  is the probability of happening of an event  $B$  given that an event  $A$  has happened, then the probability that the event  $A$  and event  $B$  both happened together is as under:

$$P(AB) = P(A).P(B/A)$$

This means that the joint probability of  $A$  and  $B$  is equal to the conditional probability of  $B$  given  $A$  times the probability of  $A$ .

Similarly, we can write:

$$P(AB) = P(B).P(A/B)$$

At this point we may now introduce the concept of marginal probability also known as unconditional probability. A marginal probability refers to the probability of happening of an event not conditional on the happening of another event. For example,  $P(A)$  and  $P(B)$  are examples of marginal probabilities. The term marginal is possibly used because such (marginal) probabilities are found in the margins of a joint probability table.

The relationship between conditional, marginal and joint probabilities can be stated as under:

$$P(B/A) = \frac{P(AB)}{P(A)}$$

Where,  $P(B/A)$  = The conditional probability of event  $B$  given that event  $A$  has happened.

$P(AB)$  = The joint probability of event  $A$  and event  $B$  happening together.

$P(A)$  = The marginal probability of the happening of event  $A$ .

**Example 8.5:** What is the probability of obtaining two heads in two throws of a single coin?

**Solution:** The probability of obtaining a head in the first throw is  $1/2$ .

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The probability of obtaining a head in the second throw is also  $1/2$  (it is not affected by the first throw of the coin).

The two throws (i.e., the two events) being independent the probability of obtaining head in both of them is the product of the probability of head in the first throw and the probability of head in the second throw.

The required probability is thus:  $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

## NOTES

### 8.3 TYPES OF PROBABILITY DISTRIBUTIONS

Probability distributions can be classified as either discrete or continuous. In a **discrete probability distribution** the variable under consideration is allowed to take only a limited number of discrete values along with corresponding probabilities. The two important discrete probability distributions are, the Binomial probability distribution and the Poisson probability distribution. In a **continuous probability distribution**, the variable under consideration is allowed to take on any value within a given range.

All these probability distributions and their functions are described in this unit. Since we have discussed the types of probabilities joint, conditional and Marginal in detail in Unit 7, we shall not discuss them here.

#### 8.3.1 Binomial Distribution

**Binomial distribution** (or the Binomial probability distribution) is a widely used probability distribution concerned with a discrete random variable and as such is an example of a discrete probability distribution. The binomial distribution describes discrete data resulting from what is often called as the Bernoulli process. The tossing of a fair coin a fixed number of times is a Bernoulli process and the outcome of such tosses can be represented by the binomial distribution. The name of Swiss mathematician Jacob Bernoulli is associated with this distribution. This distribution applies in situations where there are repeated trials of any experiment for which only one of two mutually exclusive outcomes (often denoted as ‘success’ and ‘failure’) can result on each trial.

#### The Bernoulli Process

Binomial distribution is considered appropriate in a Bernoulli process which has the following characteristics:

- (a) *Dichotomy*. This means that each trial has only two mutually exclusive possible outcomes, e.g., ‘Success’ or ‘failure’, ‘Yes’ or ‘No’, ‘Heads’ or ‘Tails’ and the like.
- (b) *Stability*. This means that the probability of the outcome of any trial is known (or given) and remains *fixed* over time, i.e., remains the same for all the trials.

- (c) *Independence*. This means that the trials are statistically independent, i.e., to say the happening of an outcome or the event in any particular trial is independent of its happening in any other trial or trials.

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### Probability Function of Binomial Distribution

The random variable, say  $X$ , in the Binomial distribution is the number of 'successes' in  $n$  trials. The probability function of the binomial distribution is written as under:

$$f(X = r) = {}^n C_r p^r q^{n-r}$$

$$r = 0, 1, 2, \dots, n$$

Where,  $n$  = Numbers of trials.

$p$  = Probability of success in a single trial.

$q = (1 - p)$  = Probability of 'failure' in a single trial.

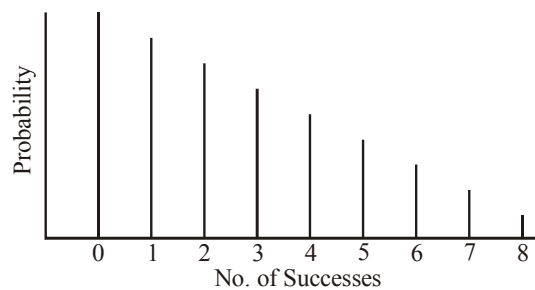
$r$  = Number of successes in ' $n$ ' trials.

### Parameters of Binomial Distribution

This distribution depends upon the values of  $p$  and  $n$  which in fact are its parameters. Knowledge of  $p$  truly defines the probability of  $X$  since  $n$  is known by definition of the problem. The probability of the happening of exactly  $r$  events in  $n$  trials can be found out using the above stated binomial function.

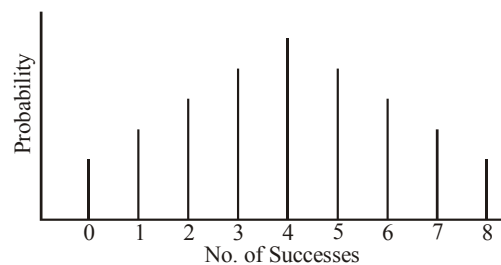
The value of  $p$  also determines the general appearance of the binomial distribution, if shown graphically. In this context the usual generalizations are:

- (a) When  $p$  is small (say 0.1), the binomial distribution is skewed to the right, i.e., the graph takes the form shown in Figure 8.1.



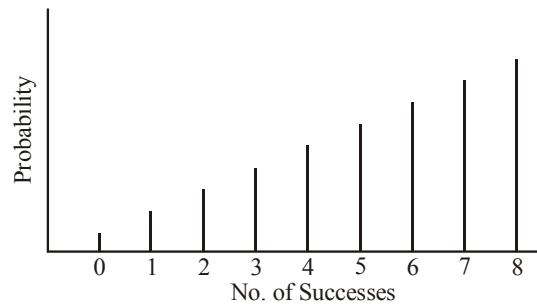
**Fig. 8.1**

- (b) When  $p$  is equal to 0.5, the binomial distribution is symmetrical and the graph takes the form as shown in Figure 8.2.



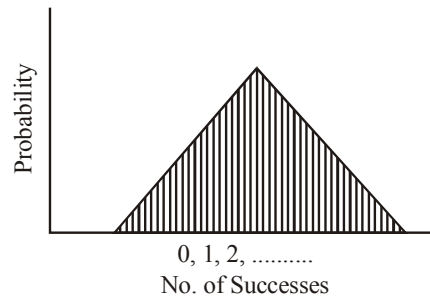
**Fig. 8.2**

(c) When  $p$  is larger than 0.5, the binomial distribution is skewed to the left and the graph takes the form as shown in Figure 8.3.



**Fig. 8.3**

But if ' $p$ ' stays constant and ' $n$ ' increases, then as ' $n$ ' increases the vertical lines become not only numerous but also tend to bunch up together to form a bell shape, i.e., the binomial distribution tends to become symmetrical and the graph takes the shape as shown in Figure 8.4.



**Fig. 8.4**

### Important Measures of Binomial Distribution

The expected value of random variable [i.e.,  $E(X)$ ] or mean of random variable (i.e.,  $\bar{X}$ ) of the binomial distribution is equal to  $n.p$  and the variance of random variable is equal to  $n.p.q$  or  $n.p.(1-p)$ . Accordingly the standard deviation of binomial distribution is equal to  $\sqrt{n.p.q}$ . The other important measures relating to binomial distribution are as under:

$$\text{Skewness} = \frac{1-2p}{\sqrt{n.p.q}}$$

$$\text{Kurtosis} = 3 + \frac{1-6p+6q^2}{n.p.q}$$

### When to Use Binomial Distribution

The use of binomial distribution is most appropriate in situations fulfilling the conditions outlined above. Two such situations, for example, can be described as follows.

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- (a) When we have to find the probability of 6 heads in 6 throws of a fair coin.
- (b) When we have to find the probability that 3 out of 10 items produced by a machine, which produces 8% defective items on an average, will be defective.

**Example 8.6:** A fair coin is thrown 10 times. The random variable  $X$  is the number of head(s) coming upwards. Using the binomial probability function, find the probabilities of all possible values which  $X$  can take and then verify that binomial distribution has a mean:  $\bar{X} = n.p$ .

and variance:  $\sigma^2 = n.p.q$

**Solution:** Since the coin is fair and so, when thrown, can come either with head upward or tail upward. Hence,  $p$  (head) =  $\frac{1}{2}$  and  $q$  (no head) =  $\frac{1}{2}$ . The required probability function is,

$$f(X = r) = {}^n C_r p^r q^{n-r}$$

$$r = 0, 1, 2 \dots 10$$

The following table of binomial probability distribution is constructed using this function.

$X_i$ (Number of Heads)	Probability $pr_i$	$X_i pr_i$	$(X_i - \bar{X})$	$(X_i - \bar{X})^2 (X_i - \bar{X})^2 . p_i$
0	${}^{10}C_0 p^0 q^{10} = 1/1024$	0/1024	-5	25
1	${}^{10}C_1 p^1 q^9 = 10/1024$	10/1024	-4	16
2	${}^{10}C_2 p^2 q^8 = 45/1024$	90/1024	-3	9
3	${}^{10}C_3 p^3 q^7 = 120/1024$	360/1024	-2	4
4	${}^{10}C_4 p^4 q^6 = 210/1024$	840/1024	-1	1
5	${}^{10}C_5 p^5 q^5 = 252/1024$	1260/1024	0	0
6	${}^{10}C_6 p^6 q^4 = 210/1024$	1260/1024	1	1
7	${}^{10}C_7 p^7 q^3 = 120/1024$	840/1024	2	4
8	${}^{10}C_8 p^8 q^2 = 45/1024$	360/1024	3	9
9	${}^{10}C_9 p^9 q^1 = 10/1024$	90/1024	4	16
10	${}^{10}C_{10} p^{10} q^0 = 1/1024$	10/1024	5	25
		$\Sigma \bar{X} = 5120/1024$	Variance = $\sigma^2 =$	
		$\bar{X} = 5$	$\Sigma (X_i - \bar{X})^2 . p_i =$	
			2560/1024 = 2.5	

The mean of the binomial distribution is given by  $n.p. = 10 \times \frac{1}{2} = 5$  and the

variance of this distribution is equal to  $n.p.q. = 10 \times \frac{1}{2} \times \frac{1}{2} = 2.5$

These values are exactly the same as we have found them in the above table. Hence, these values stand verified with the calculated values of the two measures as shown in the table.



### Fitting a Binomial Distribution

When a binomial distribution is to be fitted to the given data, then the following procedure is adopted:

- (a) Determine the values of ‘ $p$ ’ and ‘ $q$ ’ keeping in view that  $X = n \cdot p$  and  $q = (1 - p)$ .
- (b) Find the probabilities for all possible values of the given random variable applying the binomial probability function, viz.,

$$f(X_i = r) = {}^n C_r p^r q^{n-r}$$

$$r = 0, 1, 2, \dots, n$$

- (c) Work out the expected frequencies for all values of random variable by multiplying  $N$  (the total frequency) with the corresponding probability as worked out in case (b) above.
- (d) The expected frequencies so calculated constitute the fitted binomial distribution to the given data.

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### 8.3.2 Poisson Distribution

**Poisson distribution** is also a discrete probability distribution with which is associated the name of a Frenchman, Simeon Denis Poisson who developed this distribution. This distribution is frequently used in context of Operations Research and for this reason has a great significance for management people. This distribution plays an important role in *Queuing* theory, Inventory control problems and also in Risk models.

Unlike binomial distribution, Poisson distribution cannot be deduced on purely theoretical grounds based on the conditions of the experiment. In fact, it must be based on experience, i.e., on the empirical results of past experiments relating to the problem under study. Poisson distribution is appropriate specially when probability of happening of an event is very small (so that  $q$  or  $(1-p)$  is almost equal to unity) and  $n$  is very large such that the average of series (viz.  $n \cdot p$ ) is a finite number. Experience has shown that this distribution is good for calculating the probabilities associated with  $X$  occurrences in a given time period or specified area.

The random variable of interest in Poisson distribution is number of occurrences of a given event during a given interval (interval may be time, distance, area etc.). We use capital  $X$  to represent the discrete random variable and lower case  $x$  to represent a specific value that capital  $X$  can take. The probability function of this distribution is generally written as under:

$$f(X_i = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$x = 0, 1, 2, \dots$$

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Where,  $\lambda$  = Average number of occurrences per specified interval. In other words, it is the mean of the distribution.

$e$  = 2.7183 being the basis of natural logarithms.

$x$  = Number of occurrences of a given event.

### The Poisson Process

The distribution applies in case of Poisson process which has following characteristics.

- Concerning a given random variable, the mean relating to a given interval can be estimated on the basis of past data concerning the variable under study.
- If we divide the given interval into very very small intervals we will find:
  - (a) The probability that exactly one event will happen during the very very small interval is a very small number and is constant for every other very small interval.
  - (b) The probability that two or more events will happen within a very small interval is so small that we can assign it a zero value.
  - (c) The event that happens in a given very small interval is independent, when the very small interval falls during a given interval.
  - (d) The number of events in any small interval is not dependent on the number of events in any other small interval.

### Parameter and Important Measures of Poisson Distribution

Poisson distribution depends upon the value of  $\lambda$ , the average number of occurrences per specified interval which is its only parameter. The probability of exactly  $x$  occurrences can be found out using Poisson probability function stated above. The expected value or the mean of Poisson random variable is  $\lambda$  and its variance is also  $\lambda$ . The standard deviation of Poisson distribution is,  $\sqrt{\lambda}$ .

Underlying the Poisson model is the assumption that if there are on the average  $\lambda$  occurrences per interval  $t$ , then there are on the average  $k\lambda$  occurrences per interval  $kt$ . For example, if the number of arrivals at a service counter in a given hour, has a Poisson distribution with  $\lambda = 4$ , then  $y$ , the number of arrivals at a service counter in a given 6 hour day, has the Poisson distribution  $\lambda = 24$ , i.e.,  $6 \times 4$ .

### When to Use Poisson Distribution

The use of Poisson distribution is resorted to those cases when we do not know the value of ' $n$ ' or when ' $n$ ' can not be estimated with any degree of accuracy. In fact, in certain cases it does not make many sense in asking the value of ' $n$ '. For

example, the goals scored by one team in a football match are given, it cannot be stated how many goals could not be scored. Similarly, if one watches carefully one may find out how many times the lightning flashed but it is not possible to state how many times it did not flash. It is in such cases we use Poisson distribution. The number of death per day in a district in one year due to a disease, the number of scooters passing through a road per minute during a certain part of the day for a few months, the number of printing mistakes per page in a book containing many pages, are a few other examples where Poisson probability distribution is generally used.

**Example 8.7:** Suppose that a manufactured product has 2 defects per unit of product inspected. Use Poisson distribution and calculate the probabilities of finding a product without any defect, with 3 defects and with four defects.

**Solution:** If the product has 2 defects per unit of product inspected. Hence,  $\lambda = 2$ . Poisson probability function is as follows:

$$f(X_i = x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}$$

$$x = 0, 1, 2, \dots$$

Using the above probability function, we find the required probabilities as under:

$$P(\text{without any defects, i.e., } x = 0) = \frac{2^0 \cdot e^{-2}}{0!}$$

$$= \frac{1 \cdot (0.13534)}{1} = 0.13534$$

$$P(\text{with 3 defects, i.e., } x = 3) = \frac{2^3 \cdot e^{-2}}{3!} = \frac{2 \times 2 \times 2(0.13534)}{3 \times 2 \times 1}$$

$$= \frac{0.54136}{3} = 0.18045$$

$$P(\text{with 4 defects, i.e., } x = 4) = \frac{2^4 \cdot e^{-2}}{4!} = \frac{2 \times 2 \times 2 \times 2(0.13534)}{4 \times 3 \times 2 \times 1}$$

$$= \frac{0.27068}{3} = 0.09023$$

### Fitting a Poisson Distribution

When a Poisson distribution is to be fitted to the given data, then the following procedure is adopted:

- Determine the value of  $\lambda$ , the mean of the distribution.
- Find the probabilities for all possible values of the given random variable using the Poisson probability function, viz.

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$$f(X_i = x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}$$

$$x = 0, 1, 2, \dots$$

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(c) Work out the expected frequencies as follows:

$$n.p.(X_i = x)$$

(d) The result of case (c) above is the fitted Poisson distribution to the given data.

### Poisson Distribution as an Approximation of Binomial Distribution

Under certain circumstances Poisson distribution can be considered as a reasonable approximation of Binomial distribution and can be used accordingly. The circumstances which permit all this are when 'n' is large approaching to infinity and p is small approaching to zero (n = Number of trials, p = Probability of 'success'). Statisticians usually take the meaning of large n, for this purpose, when  $n \geq 20$  and by small 'p' they mean when  $p \leq 0.05$ . In cases where these two conditions are fulfilled, we can use mean of the binomial distribution (viz., n.p.) in place of the mean of Poisson distribution (viz.,  $\lambda$ ) so that the probability function of Poisson distribution becomes as stated below:

$$f(X_i = x) = \frac{(n.p)^x \cdot e^{-(n.p)}}{x!}$$

We can explain Poisson distribution as an approximation of the Binomial distribution with the help of following example.

**Example 8.8:** Given is the following information:

- (a) There are 20 machines in a certain factory, i.e.,  $n = 20$ .
- (b) The probability of machine going out of order during any day is 0.02.

What is the probability that exactly 3 machines will be out of order on the same day? Calculate the required probability using both Binomial and Poissons Distributions and state whether Poisson distribution is a good approximation of the Binomial distribution in this case.

**Solution:** Probability as per *Poisson probability function* (using n.p in place of  $\lambda$ )

(since  $n \geq 20$  and  $p \leq 0.05$ )

$$f(X_i = x) = \frac{(n.p)^x \cdot e^{-n.p}}{x!}$$

Where,  $x$  means number of machines becoming out of order on the same day.

$$\begin{aligned} P(X_i = 3) &= \frac{(20 \times 0.02)^3 \cdot e^{-(20 \times 0.02)}}{3} \\ &= \frac{(0.4)^3 \cdot (0.67032)}{3 \times 2 \times 1} = \frac{(0.064)(0.67032)}{6} \\ &= 0.00715 \end{aligned}$$

Probability as per Binomial probability function,

$$f(X_i = r) = {}^n C_r p^r q^{n-r}$$

Where,  $n = 20$ ,  $r = 3$ ,  $p = 0.02$  and hence  $q = 0.98$

$$\begin{aligned} \therefore f(X_i = 3) &= {}^{20} C_3 (0.02)^3 (0.98)^{17} \\ &= 0.00650 \end{aligned}$$

The difference between the probability of 3 machines becoming out of order on the same day calculated using probability function and binomial probability function is just 0.00065. The difference being very very small, we can state that in the given case Poisson distribution appears to be a good approximation of Binomial distribution.

**Example 8.9:** How would you use a Poisson distribution to find approximately the probability of exactly 5 successes in 100 trials the probability of success in each trial being  $p = 0.1$ ?

**Solution:** In the question we have been given,

$$n = 100 \text{ and } p = 0.1$$

$$\therefore \lambda = n.p = 100 \times 0.1 = 10$$

To find the required probability, we can use Poisson probability function as an approximation to Binomial probability function as shown below:

$$f(X_i = x) = \frac{\lambda^x \cdot e^{-\lambda}}{x} = \frac{(n.p)^x \cdot e^{-(n.p)}}{x}$$

$$\begin{aligned} \text{or } P(5) &= \frac{10^5 \cdot e^{-10}}{5} = \frac{(100000)(0.00005)}{5 \times 4 \times 3 \times 2 \times 1} = \frac{5.00000}{5 \times 4 \times 3 \times 2 \times 1} \\ &= \frac{1}{24} = 0.042 \end{aligned}$$

### 8.3.3 Normal Distribution

Among all the probability distributions the **normal probability distribution** is by far the most important and frequently used continuous probability distribution. This is so because this distribution well fits in many types of problems. This distribution is of special significance in inferential statistics since it describes

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probabilistically the link between a statistic and a parameter (i.e., between the sample results and the population from which the sample is drawn). The name of Karl Gauss, Eighteenth century mathematician-astronomer, is associated with this distribution and in honour of his contribution, this distribution is often known as the Gaussian distribution.

The normal distribution can be theoretically derived as the limiting form of many discrete distributions. For instance, if in the binomial expansion of  $(p + q)^n$ , the value of 'n' is infinity and  $p = q = \frac{1}{2}$ , then a perfectly smooth symmetrical curve would be obtained. Even if the values of  $p$  and  $q$  are not equal but if the value of the exponent 'n' happens to be very very large, we get a curve normal probability smooth and symmetrical. Such curves are called normal probability curves (or at times known as normal curves of error) and such curves represent the normal distributions.

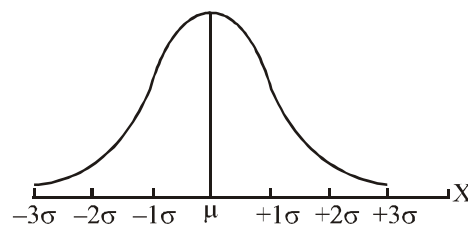
The probability function in case of normal probability distribution is given as:

$$f(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma} \right)^2}$$

Where,  $\mu$  = The mean of the distribution.

$\sigma^2$  = Variance of the distribution.

The normal distribution is thus defined by two parameters viz.,  $\mu$  and  $\sigma^2$ . This distribution can be represented graphically as shown in Figure 6.5.



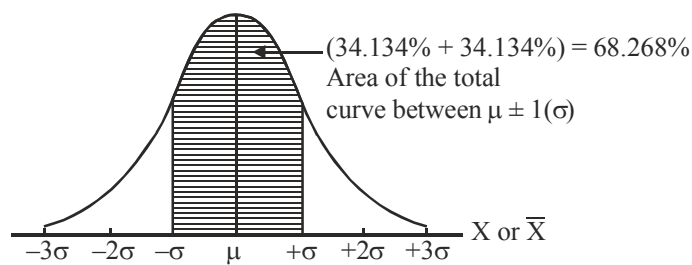
**Fig. 6.5** Curve Representing Normal Distribution

### Characteristics of Normal Distribution: Features and Applications

The characteristics of the normal distribution or that of normal curve are as given below:

1. It is symmetric distribution.
2. The mean  $\mu$  defines where the peak of the curve occurs. In other words, the ordinate at the mean is the highest ordinate. The height of the ordinate at a distance of one standard deviation from mean is 60.653% of the height of the mean ordinate and similarly the height of other ordinates at various standard

- deviations ( $\sigma$ ) from mean happens to be a fixed relationship with the height of the mean ordinate.
- The curve is asymptotic to the base line which means that it continues to approach but never touches the horizontal axis.
  - The variance ( $\sigma^2$ ) defines the spread of the curve.
  - Area enclosed between mean ordinate and an ordinate at a distance of one standard deviation from the mean is always 34.134% of the total area of the curve. It means that the area enclosed between two ordinates at one sigma (S.D.) distance from the mean on either side would always be 68.268% of the total area. This can be shown as follows:



Similarly, the other area relationships are as follows:

<i>Between</i>		<i>Area Covered to Total Area of the Normal Curve</i>
$\mu \pm 1$	S.D.	68.27%
$\mu \pm 2$	S.D.	95.45%
$\mu \pm 3$	S.D.	99.73%
$\mu \pm 1.96$	S.D.	95%
$\mu \pm 2.578$	S.D.	99%
$\mu \pm 0.6745$	S.D.	50%

- The normal distribution has only one mode since the curve has a single peak. In other words, it is always a unimodal distribution.
- The maximum ordinate divides the graph of normal curve into two equal parts.
- In addition to all the above stated characteristics the curve has the following properties:
  - $\mu = \bar{x}$
  - $\mu_2 = \sigma^2 = \text{Variance}$
  - $\mu_4 = 3\sigma^4$
  - Moment Coefficient of Kurtosis = 3

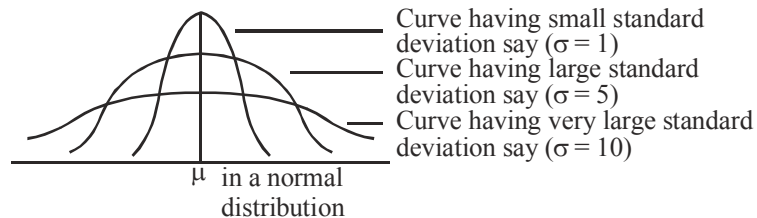
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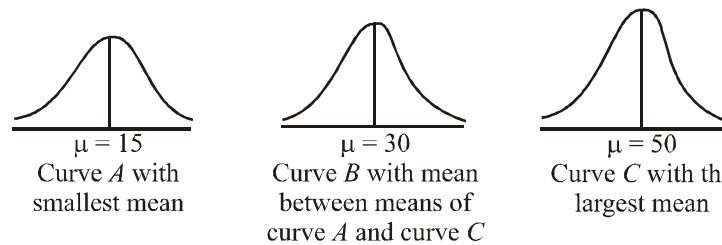
**Family of Normal Distributions**

We can have several normal probability distributions but each particular normal distribution is being defined by its two parameters viz., the mean ( $\mu$ ) and the standard deviation ( $\sigma$ ). There is, thus, not a single normal curve but rather a family of normal curves. We can exhibit some of these as under:

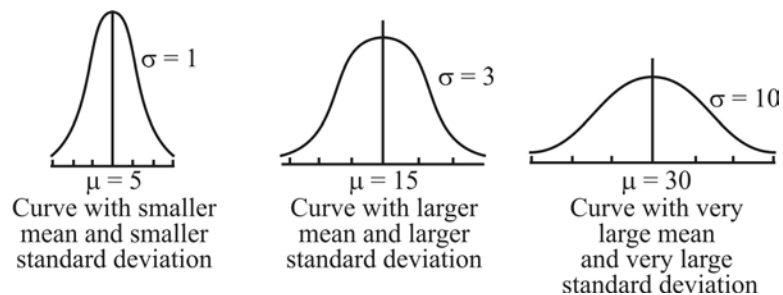
*Normal curves with identical means but different standard deviations:*



*Normal curves with identical standard deviation but each with different means:*



*Normal curves each with different standard deviations and different means:*



**How to Measure the Area under the Normal Curve?**

We have stated above some of the area relationships involving certain intervals of standard deviations (plus and minus) from the means that are true in case of a normal curve. But what should be done in all other cases? We can make use of the statistical tables constructed by mathematicians for the purpose. Using these tables we can find the area (or probability, taking the entire area of the curve as equal to 1) that the normally distributed random variable will lie within certain distances from the mean. These distances are defined in terms of standard deviations. While using the tables showing the area under the normal curve we talk in terms of standard variate (symbolically  $Z$ ) which really means standard deviations without units of measurement and this ' $Z$ ' is worked out as under:

$$Z = \frac{X - \mu}{\sigma}$$



Where,  $Z$  = The standard variate (or number of standard deviations from  $X$  to the mean of the distribution).

$X$  = Value of the random variable under consideration.

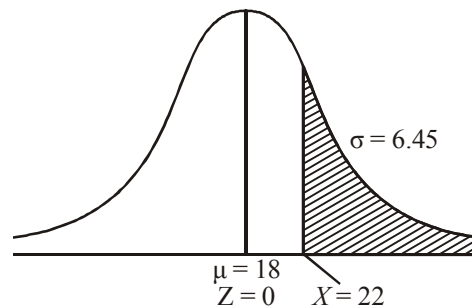
$\mu$  = Mean of the distribution of the random variable.

$\sigma$  = Standard deviation of the distribution.

The table showing the area under the normal curve (often termed as the standard normal probability distribution table) is organized in terms of standard variate (or  $Z$ ) values. It gives the values for only half the area under the normal curve, beginning with  $Z=0$  at the mean. Since the normal distribution is perfectly symmetrical the values true for one half of the curve are also true for the other half. We now illustrate the use of such a table for working out certain problems.

**Example 8.10:** A banker claims that the life of a regular saving account opened with his bank averages 18 months with a standard deviation of 6.45 months. Answer the following: (a) What is the probability that there will still be money in 22 months in a savings account opened with the said bank by a depositor? (b) What is the probability that the account will have been closed before two years?

**Solution:** (a) For finding the required probability we are interested in the area of the portion of the normal curve as shaded and shown below:



Let us calculate  $Z$  as under:

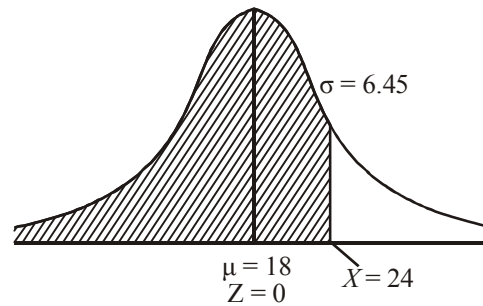
$$Z = \frac{X - \mu}{\sigma} = \frac{22 - 18}{6.45} = 0.62$$

The value from the table showing the area under the normal curve for  $Z=0.62$  is 0.2324. This means that the area of the curve between  $\mu = 18$  and  $X = 22$  is 0.2324. Hence, the area of the shaded portion of the curve is  $(0.5) - (0.2324) = 0.2676$  since the area of the entire right hand portion of the curve always happens to be 0.5. Thus the probability that there will still be money in 22 months in a savings account is 0.2676.

(b) For finding the required probability we are interested in the area of the portion of the normal curve as shaded and shown in figure:

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For the purpose we calculate,

$$Z = \frac{24 - 18}{6.45} = 0.93$$

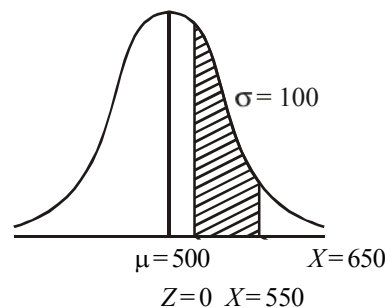
The value from the concerning table, when  $Z = 0.93$ , is 0.3238 which refers to the area of the curve between  $\mu = 18$  and  $X = 24$ . The area of the entire left hand portion of the curve is 0.5 as usual.

Hence, the area of the shaded portion is  $(0.5) + (0.3238) = 0.8238$  which is the required probability that the account will have been closed before two years, i.e., before 24 months.

**Example 8.11:** Regarding a certain normal distribution concerning the income of the individuals we are given that mean = 500 rupees and standard deviation = 100 rupees. Find the probability that an individual selected at random will belong to income group,

- (a) ₹ 550 to ₹ 650; (b) ₹ 420 to 570.

**Solution:** (a) For finding the required probability we are interested in the area of the portion of the normal curve as shaded and shown below:



For finding the area of the curve between  $X = 550$  to 650, let us do the following calculations:

$$Z = \frac{550 - 500}{100} = \frac{50}{100} = 0.50$$

Corresponding to which the area between  $\mu = 500$  and  $X = 550$  in the curve as per table is equal to 0.1915 and,

$$Z = \frac{650 - 500}{100} = \frac{150}{100} = 1.5$$

Corresponding to which the area between  $\mu = 500$  and  $X = 650$  in the curve as per table is equal to 0.4332

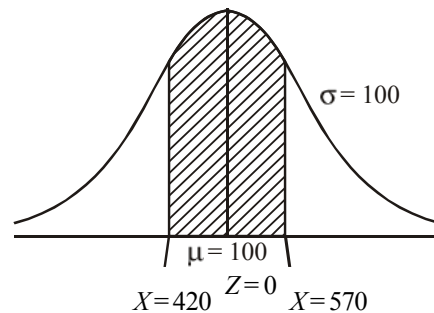
Hence, the area of the curve that lies between  $X = 550$  and  $X = 650$  is,

$$(0.4332) - (0.1915) = 0.2417$$

This is the required probability that an individual selected at random will belong to income group of ₹ 550 to ₹ 650.

(b) For finding the required probability we are interested in the area of the portion of the normal curve as shaded and shown below:

To find the area of the shaded portion we make the following calculations:



$$Z = \frac{570 - 500}{100} = 0.70$$

Corresponding to which the area between  $\mu = 500$  and  $X = 570$  in the curve as per table is equal to 0.2580.

and  $Z = \frac{420 - 500}{100} = -0.80$

Corresponding to which the area between  $\mu = 500$  and  $X = 420$  in the curve as per table is equal to 0.2881.

Hence, the required area in the curve between  $X = 420$  and  $X = 570$  is,

$$(0.2580) + (0.2881) = 0.5461$$

This is the required probability that an individual selected at random will belong to income group of ₹ 420 to ₹ 570.

### Check Your Progress

1. List the types of probability distributions.
2. List the various types of discrete and continuous probability distributions.
3. Why is Bernoulli process considered appropriate?
4. Write the probability function of binomial distribution.
5. Under what condition is poisson distribution used?

### NOTES

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## 8.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

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### NOTES

1. There are two types of probability distributions, discrete and continuous probability distributions. In discrete probability distribution, the variable under consideration is allowed to take only a limited number of discrete values along with corresponding probabilities. On the other hand, in a continuous probability distribution, the variable under consideration is allowed to take on any value within a given range.
2. There are two types of discrete probability distributions, binomial probability distribution and Poisson probability distribution. The continuous probability distributions, is also of two type, i.e., exponential probability distribution and normal probability distribution.
3. Bernoulli process or Binomial distribution is considered appropriate and has the following characteristics;
  - (a) Dichotomy: This means that each trial has only two mutually exclusive possible outcomes. For example, success or failure, yes or no, heads or tail, etc.
  - (b) Stability: This means that the probability of the outcome of any trial is known and remains fixed over time, i.e. remains the same for all the trials.
  - (c) Independence: This means that the trials are statistically independent, i.e., to say the happening of an outcome or the event in any particular trial is independent of its happening in any other trial or trials.
4. The probability function of binomial distribution is written as under:

$$f(X=r) = {}^n C_r p^r q^{n-r}$$
$$r = 0, 1, 2, \dots, n$$

Where,  $n$  = Numbers of trials.

$p$  = Probability of success in a single trial.

$q = (1 - p)$  = Probability of failure in a single trial.

$r$  = Number of successes in  $n$  trials.

5. Poisson distribution is used when the probability of happening of an event is very small and  $n$  is very large such that the average of series is a finite number. This distribution is good for calculating the probabilities associated with  $X$  occurrences in a given time period or specified area.

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## 8.5 SUMMARY

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- Binomial distribution is probably the best known of discrete distributions. The normal distribution or  $Z$ -distribution, is often used to approximate the binomial distribution.

- However, if the sample size is very large, the Poisson distribution is a philosophically more correct alternative to binomial distribution than normal distribution.
- One of the main differences between the Poisson distribution and the binomial distribution is that in using the binomial distribution all eligible phenomena are studied, whereas in the Poisson only the cases with a particular outcome are studied.
- Amongst all, the normal probability distribution is by far the most important and frequently used distribution because it fits well in many types of problems.

## NOTES

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### 8.6 KEY WORDS

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- **Discrete probability distribution:** In a discrete probability distribution the variable under consideration is allowed to take only a limited number of discrete values along with corresponding probabilities
- **Binomial distribution:** It is also called as Bernoulli process and is used to describe discrete random variable
- **Poisson distribution:** It is used to describe the empirical results of past experiments relating to the problem and plays important role in queuing theory, inventory control problems and risk models

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### 8.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

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#### Short Answer Questions

1. Define probability distribution and probability functions.
2. Describe binomial distribution and its measures.
3. How an binomial distribution be fitted to a given data?
4. Describe Poisson distribution and its important measures.
5. Poisson distribution can be an approximation of binomial distribution. Explain.

#### Long Answer Questions

1. (a) Explain the meaning of Bernoulli process pointing out its main characteristics.  
(b) Give a few examples narrating some situations wherein binomial probability distribution can be used.
2. State the distinctive features of the Binomial, Poisson and Normal probability distributions. When does a Binomial distribution tend to become a Normal and a Poisson distribution? Explain.

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3. Explain the circumstances when the following probability distributions are used:

- (a) Binomial distribution
- (b) Poisson distribution
- (c) Exponential distribution
- (d) Normal distribution

4. The following mistakes per page were observed in a book:

<i>No. of Mistakes Per Page</i>	<i>No. of Times the Mistake Occurred</i>
0	211
1	90
2	19
3	5
4	0
Total	
345	

Fit a Poisson distribution to the data given above and test the goodness of fit.

5. Fit a normal distribution to the following data:

<i>Height in Inches</i>	<i>Frequency</i>
60–62	5
63–65	18
66–68	42
69–71	27
72–74	8

**8.8 FURTHER READINGS**

Taha, H.A. 2006. *Operations Research: An Introduction*, 8th edition. New Delhi: Prentice-Hall of India.

Jensen, Paul A., and Jonathan F. Bard. 2003. *Operations Research Models and Methods*. New York: John Wiley & Sons.

Gupta, P.K., and D.S. Hira. 2002. *Introduction to Operations Research*. New Delhi: S. Chand & Co.

Kothari, C.R. 1982. *An Introduction to Operational Research*. New Delhi: Vikas Publishing House Pvt. Ltd.

Kalavathy, S. 2013. *Operations Research*, 4th edition. New Delhi: Vikas Publishing House Pvt. Ltd.

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**BLOCK - III  
OPERATIONAL RESEARCH AND  
SIMULATION TECHNIQUES**

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**NOTES**

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**UNIT 9 OPERATIONS RESEARCH  
FOR DECISION MAKING**

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**Structure**

- 9.0 Introduction
- 9.1 Objectives
- 9.2 Operations Research: Definition, Background and Developments
  - 9.2.1 Nature and Characteristics of Quantitative Methods
  - 9.2.2 Development of Operational Research
- 9.3 Operations Research as a Tool in Decision-Making
  - 9.3.1 Benefits of Quantitative Methods
- 9.4 Phases of Operations Research Study
  - 9.4.1 Models in Operations Research and Methods of Deriving the Solution
  - 9.4.2 Limitations of Operations Research
- 9.5 Answers to Check Your Progress Questions
- 9.6 Summary
- 9.7 Key Words
- 9.8 Self Assessment Questions and Exercises
- 9.9 Further Reading

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**9.0 INTRODUCTION**

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In this unit, you will learn about Operations Research (OR), which is a very powerful tool for decision-making. The term Operations Research was coined by J.F. McCloskey and F.N. Trefethen in 1940 in Bowdsey in the United Kingdom. This innovative science was discovered during World War II for a specific military situation, when military management sought decisions based on the optimal consumption of limited military resources with the help of an organized and systematized scientific approach. This was termed as operations research or operational research. Thus, OR was known as an ability to win a war without really going into a battlefield or fighting it. It is a new field of scientific and managerial application.

The different phases of applications include—implementing and formulating the problem, constructing a mathematical model, deriving the result from the model, testing and updating the model and controlling the final output or solution. You will learn about the requirements of a good model which must be capable of working

on a new formulation without making any changes in its frame with minimum assumptions and with minimum variables and must not take extraordinary time to solve the problem.

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### 9.1 OBJECTIVES

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After going through this unit, you will be able to:

- Understand the definition and scope of operations research (OR)
- Define development of OR
- Analyse the nature and characteristics of OR
- Describe OR as a tool in decision-making
- Explain the relation between OR and management
- Know about the phases and applications of OR
- Understand the requirements of a good model
- Discuss the benefits and limitations of quantitative methods

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### 9.2 OPERATIONS RESEARCH: DEFINITION, BACKGROUND AND DEVELOPMENTS

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The term, operations research was first coined in 1940 by J.F. McCloskey and F.N. Trefethen in a small town Bowdsey in the United Kingdom. This new science came into existence in a military context. During World War II, military management called on scientists from various disciplines and organized them into teams to assist in solving strategic and tactical problems, relating to air and land defence. Their mission was to formulate specific proposals and plans for aiding the Military commands to arrive at decisions on optimal utilization of scarce military resources and attempts to implement these decisions effectively. This new approach to the systematic and scientific study of the operations of the system was called Operations Research (OR) or Operational Research. Hence, OR can be associated with ‘an art of winning the war without actually fighting it’.

#### Definitions

Operations Research (OR) has been defined so far in various ways and it is perhaps still too young to be defined in some authoritative way. It is not possible to give uniformly acceptable definitions of OR. A few opinions about the definition of OR are given below. These have been changed according to the development of the subject.

OR is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control.

*Morse and Kimball (1946)*



OR is the scientific method of providing executive with an analytical and objective basis for decisions.

*P.M.S. Blackett (1948)*

OR is a systematic method-oriented study of the basic structures, characteristics, functions and relationships of an organization to provide the executive with a sound, scientific and quantitative basis for decision-making.

*E.L. Arnoff and M.J Netzorg*

OR is a scientific approach to problem solving for executive management.

*H.M. Wagner*

OR is an aid for the executive in making his decisions by providing him with the quantitative information based on the scientific method of analysis.

*C.Kittee*

OR is the scientific knowledge through interdisciplinary team effort for the purpose of determining the best utilization of limited resources.

*H.A. Taha*

The various definitions given here bring out the following essential characteristics of operations research:

- (i) System orientation
- (ii) Use of interdisciplinary terms
- (iii) Application of scientific methods
- (iv) Uncovering new problems
- (v) Quantitative solutions
- (vi) Human factors

### **Scope of Operations Research**

There is a great scope for economists, statisticians, administrators and the technicians working as a team to solve problems of defence by using the OR approach. Besides this, OR is useful in various other important fields like:

- (i) Agriculture
- (ii) Finance
- (iii) Industry
- (iv) Marketing
- (v) Personnel management
- (vi) Production management
- (vii) Research and development

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### Phases of Operations Research

The procedure to be followed in the study of OR generally involves the following major phases:

- (i) Formulating the problem
- (ii) Constructing a mathematical model
- (iii) Deriving the solution from the model
- (iv) Testing the model and its solution (updating the model)
- (v) Controlling the solution
- (vi) Implementation

#### 9.2.1 Nature and Characteristics of Quantitative Methods

Looking to the basic features of the definitions concerning OR, we can state that, ‘Operational Research can be considered as the application of scientific method by interdisciplinary teams to problems involving the control of organized (man-machine) systems to provide solutions, which best serve the purposes of the organization as a whole’.

Different characteristics constituting the nature of OR and quantitative methods can be summed up as follows:

1. *Interdisciplinary team approach*: The Operations Research has the characteristics that it is done by a team of scientists drawn from various disciplines such as mathematics, statistics, economics, engineering, physics, etc. It is essentially an interdisciplinary team approach. Each member of the OR team is benefited from the viewpoint of the other so that a workable solution obtained through such collaborative study has a greater chance of acceptance by management.
2. *Systems approach*: Operations Research emphasizes on the overall approach to the system. This characteristic of OR is often referred as system orientation. The orientation is based on the observation that in the organized systems the behaviour of any part ultimately has some effect on every other part. But all these effects are not significant and even not capable of detection. Therefore, the essence of system orientation lies in the systematic search for significant interactions in evaluating actions of any part of the organization. In OR an attempt is made to take account of all the significant effects and to evaluate them as a whole. OR thus considers the total system for getting the optimum decisions.
3. *Helpful in improving the quality of solution*: Operations Research cannot give perfect answers or solutions to the problems. It merely gives bad answers to the problems which otherwise have worst answers. Thus, OR simply helps in improving the quality of the solution but does not result into a perfect solution.

4. *Scientific method*: Operations Research involves scientific and systematic attack of complex problems to arrive at the optimum solution. In other words, Operations Research or OR uses techniques of scientific research. Thus OR comprehends both aspects, i.e., it includes both scientific research on the phenomena of operating systems and the associated engineering activities aimed at applying the results of research.
5. *Goal oriented optimum solution*: Operations Research tries to optimize a well-defined function subject to given constraints and as such is concerned with the optimization theory.
6. *Use of models*: Operations Research uses models built by quantitative measurement of the variables concerning a given problem and also derives a solution from the model using one or more of the diversified solution techniques. A solution may be extracted from a model either by conducting experiments on it or by mathematical analysis. The purpose is to help the management to determine its policy and actions scientifically.
7. *Require willing executives*: Operations Research does require the willingness on the part of the executive for experimentation to evaluate the costs and the consequences of the alternative solutions of the problem. It enables the decision-maker to be objective in choosing an alternative from among many possible alternatives.
8. *Reduces complexity*: Operations Research tries to reduce the complexity of business operations and does help the executive in correcting a troublesome function and to consider innovations which are too costly and complicated to experiment with the actual practice.

In view of the above, OR must be viewed as both a science and an art. As science, OR provides mathematical techniques and algorithms for solving appropriate decision problems. OR is an art because success in all the phases that precede and succeed the solution of a problem largely depends on the creativity and personal ability of the decision-making analysts.

### 9.2.2 Development of Operational Research

The subject of Operational Research (OR) was developed in military context during World War II, pioneered by the British scientists. At that time, the military management in England appointed a study group of scientists to deal with the strategic and tactical problems related to air and land defence of the country. The main reason for conducting the study was that they were having very limited military resources. It was, therefore, necessary to decide upon the most effective way of utilizing these resources. As the name implies, Operations Research was apparently invented because the team was dealing with research on military operations. The scientists studied the various problems and on the basis of quantitative study of operations suggested certain approaches which showed remarkable success. The encouraging results obtained by the British operations research teams consisting

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of personnel drawn from various fields like Mathematics, Physics, Biology, Psychology and other physical sciences, quickly motivated the United States military management to start similar activities. Successful innovations of the US teams included the development of new flight patterns, planning sea mining and effective utilization of electronic equipments. Similar OR teams also started functioning in Canada and France. These OR teams were usually assigned to the executive-in-charge of operations and as such their work came to be known as 'Operational Research' in the UK and by a variety of names in the United States an Operational Analysis, Operations Evaluation, Operations Research, Systems Analysis, Systems Evaluation and Systems Research. The name 'Operational Research' or 'Operations Research' or simply OR is most widely used now a days all over the world for the systematic and scientific study of the operations of the system. Till fifties, use of OR was mainly confined to military purposes.

After the end of the second world war, the success of military teams attracted the attention of industrial managers who were seeking solutions to their complex managerial problems. At the end of the war, expenditures on defence research were reduced in the UK and this led to the release of many operations research workers from the military at a time when industrial managers were confronted with the need to reconstruct most of Britain's manufacturing industries and plants that had been damaged in war. Executives in such industries sought assistance from the said operations research workers. But in the USA most of the war experienced operations research workers remained in military service as the defence research was increased and consequently operations research was expanded at the end of the war. It was only in the early 1950s, industry in the USA began to absorb the operations is research worker under the pressure for increased demands for greater productivity originated because of the outbreak of the Korean conflict and because of technological developments in industry. Thus, OR began to develop in industrial field in the United States since the year 1950. The Operations Research Society of America was formed in 1953 and in 1957 the International Federation of Operational Research Societies was established. Various journals relating to OR began to appear in different countries in the years that followed the mid-fifties. Courses and curricula in OR in different universities and other academic institutions began to proliferate in the United States. Other countries rapidly followed this and after the late fifties Operations Research was applied for solving business and industrial problems. Introduction of Electronic Data Processing (EDP) methods further enlarged the scope for application of OR techniques. With the help of a digitized computer many complex problems can be studied on day-to-day basis. As a result, many industrial concerns are adopting OR as an integrated decision-making tool for their routine decision procedures.

### **Operations Research in India**

Today, the impact of OR in Indian business and industry can be felt in many areas. A large number of management consulting firms are recently engaged in OR

activities. Apart from military and business applications, OR activities include transportation systems, libraries, hospitals, city planning, financial institutions, etc. With increasing use of computers the operations research techniques have started playing a noticeable role in our country as well. The major Indian industries such as Delhi Cloth Mills, Indian Railways, Indian Airlines, Defence Organizations, Hindustan Liver, Tata Iron and Steel Company, Fertilizer Corporation of India and similar industries make use of operations research techniques for solving problems and making decisions.

Historically, Operations Research started developing in India after independence specially with the setting up of an Operation Research Unit at the Regional Research Laboratory at Hyderabad in 1949. Operations Research activities gained further impetus with the establishment of an Operations Research Unit in 1953 in the Indian Statistical Institute (ISI), Calcutta for applying Operations Research techniques in national planning and survey. Operational Research Society of India was formed in 1957 which joined International Federation of Operational Research Societies in 1960 by becoming its member. The said society helped the cause of the development of Operations Research activities in India in several ways and started publishing a journal of Operations Research entitled 'OPSEARCH' from 1963. Besides, the Indian Institute of Industrial Engineers has also promoted Operations Research in India and its journals viz., 'Industrial Engineering' and 'Management' are considered as important key journals relating to Operations Research in the country. Other important journals which deal with Operations Research in our country are the Journal of the National Productivity Council, Materials Management Journal of India and the Defence Science Journal. There are several institutions which train and produce people in the field of Operations Research to meet the need of OR practitioners in the country.

So far as the application of Operations Research in India is concerned it was Professor P.C. Mahalonobis of ISI, Calcutta who made the first important application. He formulated the Second Five Year Plan of our country with the help of OR technique to forecast the trends of demand, availability of resources and for scheduling the complex scheme necessary for developing our country's economy. It was estimated that India could become self-sufficient in food and solve her foreign exchange problems merely by reducing the wastage of food by 15%. Operational Research Commission made the use of OR techniques for planning the optimum size of the Caravelle fleet of Indian Airlines. Kirloskar company made use of assignments models for allocation of their salesmen to different areas so as to maximize their profit. Linear Programming (LP) models were also used by them to assemble various diesel engines at minimum possible cost. Various cotton textile leaders such as Binny, DCM, Calico, etc., are using linear programming techniques in cotton blending. Many other firms like Union Carbide, ICI, TELCO and Hindustan Liver etc., are making use of OR techniques for solving many of their complex business problems. State Trading Corporation of India (STCI) has also set up a Management Sciences Group with the idea of promoting and developing

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the use of OR techniques in solving its management decision problems. Besides, many Universities and professional academic institutions are imparting training in OR in our country. The subject of OR has been included in the courses of such institutions. But in comparison with the western world the present state of OR in our country is much behind. Operations Research activities are very much limited and confined only to the big organized industries. Most popular practical application of Operations Research has been mainly that of Linear Programming. There is relative scarcity of well-trained operational researchers. The use of Operations Research is relatively a very costly affair. In spite of several limitations, our industrialists are gradually becoming conscious of the role of Operations Research techniques and in the coming years such techniques will have an increasingly important role to play in Indian business and industry.

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### 9.3 OPERATIONS RESEARCH AS A TOOL IN DECISION-MAKING

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Mathematical models have been constructed for OR problems and methods for solving the models are available in many cases. Such methods are usually termed as OR techniques. Some of the important OR techniques often used by decision-makers in modern times in business and industry are as under:

1. *Linear programming.* This technique is used in finding a solution for optimizing a given objective, such as profit maximization or cost minimization under certain constraints. This technique is primarily concerned with the optimal allocation of limited resources for optimizing a given function. The name linear programming is given because of the fact that the model in such cases consists of linear equations indicating linear relationship between the different variables of the system. Linear programming technique solves product-mix and distribution problems of business and industry. It is a technique used to allocate scarce resources in an optimum manner in problems of scheduling, product-mix and so on. Key factors under this technique include an objective function, choice among several alternatives, limits or constraints stated in symbols and variables assumed to be linear.
2. *Waiting line or Queuing theory.* Waiting line or queuing theory deals with mathematical study of queues. The queues are formed whenever the current demand for service exceeds the current capacity to provide that service. Waiting line technique concerns itself with the random arrival of customers at a service station where the facility is limited. Providing too much of capacity will mean idle time for servers and will lead to waste of 'money'. On the other hand, if the queue becomes long there will be a cost due to waiting of units in the queue. Waiting line theory, therefore, aims at minimizing the costs of both servicing and waiting. In other words, this technique is used to analyse the feasibility of adding facilities and to assess the amount and cost of waiting time. With its help we can find the optimal capacity to be installed

which will lead to a sort of an economic balance between cost of service and cost of waiting.

3. *Inventory control/Planning.* Inventory planning aims at optimizing inventory levels. Inventory may be defined as a useful idle resource which has economic value, e.g., raw materials, spare parts, finished products, etc. Inventory planning, in fact, answers the two questions, viz., how much to buy and when to buy. Under this technique the main emphasis is on minimizing costs associated with holding of inventories, procurement of inventories and the shortage of inventories.
4. *Game theory.* Game theory is used to determine the optimum strategy in a competitive situation. Simplest possible competitive situation is that of two persons playing zero-sum game, i.e., a situation in which two persons are involved and one person wins exactly what the other loses. More complex competitive situations in real life can be imagined where game theory can be used to determine the optimum strategy.
5. *Decision theory.* Decision theory concerns with making sound decisions under conditions of certainty, risk and uncertainty. As a matter of fact there are three different kinds of states under which decisions are made, viz., deterministic, stochastic and uncertainty and the decision theory explains how to select a suitable strategy to achieve some object or goal under each of these three states.
6. *Network analysis.* Network analysis involves the determination of an optimum sequence of performing certain operations concerning some jobs in order to minimize overall time and/or cost. Programme Evaluation and Review Technique (PERT), Critical Path Method (CPM) and other network techniques such as Gantt Chart come under network analysis. Key concepts under this technique are network of events and activities, resource allocation, time and cost considerations, network paths and critical paths.
7. *Simulation.* Simulation is a technique of testing a model, which resembles a real life situation. This technique is used to imitate an operation prior to actual performance. Two methods of simulation are used—Monte Carlo method and the System simulation method. The former uses random numbers to solve problems which involve conditions of uncertainty and the mathematical formulation is impossible. In case of System simulation, there is a reproduction of the operating environment and the system allows for analysing the response from the environment to alternative management actions. This method draws samples from a real population instead of from a table of random numbers.
8. *Integrated production models.* This technique aims at minimizing cost with respect to work force, production and inventory. This technique is highly complex and is used only by big business and industrial units. This technique can be used only when sales and costs statistics for a considerable long period are available.

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9. *Some other OR techniques.* In addition, there are several other techniques such as non-linear programming, dynamic programming, search theory, the theory of replacement, etc. A brief mention of some of these is as follows:

- (i) *Non-linear programming.* A form of programming in which some or all of the variables are curvilinear. In other words, this means that either the objective function or constraints or both are not in linear form. In most of the practical situations, we encounter non-linear programming problems but for computation purpose we approximate them as linear programming problems. Even then there may remain some non-linear programming problems which may not be fully solved by presently known methods.
- (ii) *Dynamic programming.* It refers to a systematic search for optimal solutions to problems that involve many highly complex interrelations that are, moreover, sensitive to multistage effects such as successive time phases.
- (iii) *Heuristic programming.* It is also known as discovery method and refers to step-by-step search towards an optimum when a problem cannot be expressed in mathematical programming form. The search procedure examines successively a series of combinations that lead to stepwise improvements in the solution and the search stops when a near optimum has been found.
- (iv) *Integer programming.* It is a special form of linear programming in which the solution is required in terms of integral numbers (i.e., whole numbers) only.
- (v) *Algorithmic programming.* It is just the opposite of Heuristic programming. It may also be termed as similar to mathematical programming. This programming refers to a thorough and exhaustive mathematical approach to investigate all aspects of the given variables in order to obtain optimal solution.
- (vi) *Quadratic programming.* It refers to a modification of linear programming in which the objective equations appear in quadratic form, i.e., they contain squared terms.
- (vii) *Parametric programming.* It is the name given to linear programming when it is modified for the purpose of inclusion of several objective equations with varying degrees of priority. The sensitivity of the solution to these variations is then studied.
- (viii) *Probabilistic programming.* It is also known as stochastic programming and refers to linear programming that includes an evaluation of relative risks and uncertainties in various alternatives of choice for management decisions.
- (ix) *Search theory.* It concerns itself with search problems. A search problem is characterized by the need for designing a procedure to



collect information on the basis of which one or more decisions are made. This theory is useful in places in which some events are known to occur but the exact location is not known. The first search model was developed during World War II to solve decision problems connected with air patrols and their search for submarines. Advertising agencies search for customers, personnel departments search for good executives are some of the examples of search theory's application in business.

- (x) *The theory of replacement.* It is concerned with the prediction of replacement costs and determination of the most economic replacement policy. There are two types of replacement models. First type of models deal in replacing equipments that deteriorate with time and the other type of models help in establishing replacement policy for those equipments which fail completely and instantaneously.

All these techniques are not simple but involve higher mathematics. The tendency today is to combine several of these techniques and form into more sophisticated and advanced programming models.

### 9.3.1 Benefits of Quantitative Methods

Operational research renders valuable service in the field of business management. It ensures improvement in the quality of managerial decisions in all functional areas of management. The role of OR in business management can be summed up as under:

*OR techniques help the directing authority* in optimum allocation of various limited resources, viz., men, machines, money, material, time, etc., to different competing opportunities on an objective basis for achieving effectively the goal of a business unit. They help the chief of executive in broadening management vision and perspectives in the choice of alternative strategies to the decision problems such as forecasting manpower, production capacities, capital requirements and plans for their acquisition.

*OR is useful to the production management* in (i) Selecting the building site for a plant, scheduling and controlling its development and designing its layout; (ii) Locating within the plant and controlling the movements of required production materials and finished goods inventories; (iii) Scheduling and sequencing production by adequate preventive maintenance with optimum number of operatives by proper allocation of machines; and (iv) Calculating the optimum product-mix.

*OR is useful to the personnel management* to find out (i) Optimum manpower planning; (ii) The number of persons to be maintained on the permanent or full time roll; (iii) The number of persons to be kept in a work pool intended for meeting absenteeism; (iv) The optimum manner of sequencing and routing of personnel to a variety of jobs; and (v) In studying personnel recruiting procedures, accident rates and labour turnover.

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*OR techniques equally help the marketing management* to determine (i) Where distribution points and warehousing should be located; their size, quantity to be stocked and the choice of customers; (ii) The optimum allocation of sales budget to direct selling and promotional expenses; (iii) The choice of different media of advertising and bidding strategies; and (iv) The consumer preferences relating to size, colour, packaging, etc., for various products as well as to outbid and outwit competitors.

*OR is also very useful to the financial management* in (i) Finding long-range capital requirements as well as how to generate these requirements; (ii) Determining optimum replacement policies; (iii) Working out a profit plan for the firm; (iv) Developing capital-investments plans; and (v) Estimating credit and investment risks.

In addition to all this, OR provides the business executives such an understanding of the business operations which gives them new insights and capability to determine better solutions for several decision-making problems with great speed, competence and confidence. When applied on the level of management where policies are formulated, OR assists the executives in an advisory capacity but on the operational level where production, personnel, purchasing, inventory and administrative decisions are made. It provides management with a means for handling and processing information. Thus, in brief, OR can be considered as scientific method of providing executive departments with a quantitative basis for taking decisions regarding operations under their control.

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### 9.4 PHASES OF OPERATIONS RESEARCH STUDY

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OR study generally involves three phases viz., the judgement phase, the research phase and the action phase. Of these three, the research phase is the longest and the largest, but the remaining two phases are very important since they provide the basis for and implementation of the research phase respectively.

*The judgement phase* includes (i) A determination of the problem; (ii) The establishment of the objectives and values related to the operation; and (iii) The determination of suitable measures of effectiveness.

*The research phase* utilizes (i) Observations and data collection for better understanding of the problem; (ii) Formulation of hypothesis and models; (iii) Observation and experimentation to test the hypothesis on the basis of additional data; and (iv) predictions of various results from the hypothesis, generalization of the result and consideration of alternative methods.

*The action phase* in the OR consists of making recommendations for decision process. As such this phase deals with the implementation of the tested results of the model. This phase is executed primarily through the cooperation of the OR experts on the one hand and those who are responsible for operating the system on the other.

## Methodology of Operations Research

In view of the above referred phases the methodology of OR generally involves the following steps:

1. *Formulating the problem.* The first step in an OR study is to formulate the problem in an appropriate form. Formulating a problem consists in identifying, defining and specifying the measures of the components of a decision model. This means that all quantifiable factors which are pertinent to the functioning of the system under consideration are defined in mathematical language as variables (factors which are not controllable), parameters or coefficient along with the constraints on the variables and the determination of suitable measures of effectiveness.
2. *Constructing the model.* The second step consists in constructing the model by which we mean that appropriate mathematical expressions are formulated which describe interrelations of all variables and parameters. In addition, one or more equations or inequalities are required to express the fact that some or all of the controlled variables can only be manipulated within limits. Such equations or inequalities are termed as constraints or the restrictions. The model must also include an objective function which defines the measure of effectiveness of the system. The objective function and the constraints, together constitute a model of the problem that we want to solve. This model describes the technology and the economics of the system under consideration through a set of simultaneous equations and inequalities.
3. *Deriving the solution.* Once the model is constructed the next step in an OR study is that of obtaining the solution to the model, i.e., finding the optimal values of the controlled variables—values that produce the best performance of the system for specified values of the uncontrolled variables. In other words, an optimum solution is determined on the basis of the various equations of the model satisfying the given constraints and inter-relations of the system and at the same time maximizing profit or minimizing cost or coming as close as possible to some other goal or criterion. How the solution can be derived depends on the nature of the model. In general, there are three methods available for the purpose viz., the analytical methods, the numerical methods and the simulation methods. Analytical methods involve expressions of the model by mathematical computations and the kind of mathematics required depends upon the nature of the model under consideration. This sort of mathematical analysis can be conducted only in some cases without any knowledge of the values of the variables but in others the values of the variables must be known concretely or numerically. In latter cases, we use the numerical methods which are concerned with iterative procedures through the use of numerical computations at each step. The algorithm (or the set of computational rules) is started with a trial or

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initial solution and continued with a set of rules for improving it towards optimality. The initial solution is then replaced by the improved one and the process is repeated until no further improvement is possible. But in those cases where the analytical as well as the numerical methods cannot be used for deriving the solution then we use simulation methods, i.e., we conduct experiments on the model in which we select values of the uncontrolled variables with the relative frequencies dictated by their probability distributions. The simulation methods involve the use of probability and sampling concepts and are generally used with the help of computers. Whichever method is used, our objective is to find an optimal or near-optimal solution, i.e., a solution which optimizes the measure of effectiveness in a model.

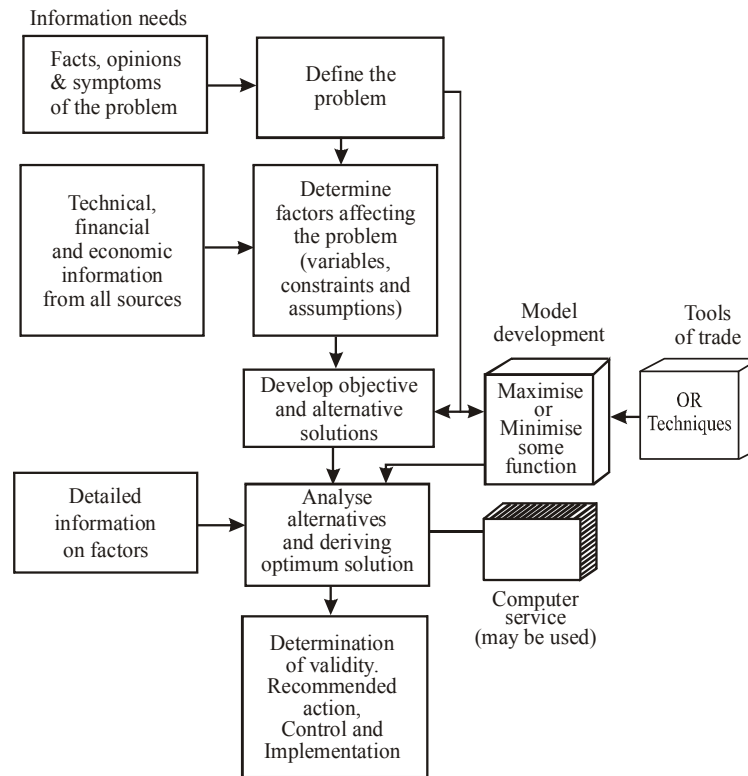
4. *Testing the validity.* The solution values of the model, obtained as stated in step three above, are then tested against actual observations. In other words, effort is made to test the validity of the model used. A model is supposed to be valid if it can give a reliable prediction of the performance of the system represented through the model. If necessary, the model may be modified in the light of actual observations and the whole process is repeated till a satisfactory model is attained. The operational researcher quite often realizes that his model must be a good representation of the system and must correspond to reality which in turn requires this step of testing the validity of the model in an OR study. In effect, performance of the model must be compared with the policy or procedure that it is meant to replace.
5. *Controlling the solution.* This step of an OR study establishes control over the solution by proper feedback of information on variables which might have deviated significantly. As such the significant changes in the system and its environment must be detected and the solution must accordingly be adjusted.
6. *Implementing the results.* Implementing the results constitutes the last step of an OR study. The objective of OR is not merely to produce reports but to improve the performance of systems. The results of the research must be implemented if they are accepted by the decision-makers. It is through this step that the ultimate test and evaluation of the research is made and it is in this phase of the study the researcher has the greatest opportunity for learning.

Thus the procedure for an OR study generally involves some major steps viz., formulating the problem, constructing the mathematical model to represent the system under study, deriving a solution from the model, testing the model and the solution so derived, establishing controls over the solution and lastly putting the solution to work-implementation. Although the said phases and the steps are usually initiated in the order listed in an OR study but it should always be kept in

mind that they are likely to overlap in time and to interact each phase usually continues until the study is completed.

### Flow Chart Showing OR Approach

OR approach can be well illustrated through the following flow chart:



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#### 9.4.1 Models in Operations Research and Methods of Deriving the Solution

A model in OR is a simplified representation of an operation or as a process in which only the basic aspects or the most important features of a typical problem under investigation are considered. The objective of a model is to identify significant factors and interrelationships. The reliability of the solution obtained from a model depends on the validity of the model representing the real system.

A good model must possess the following characteristics:

- (i) It should be capable of taking into account new formulation without having any changes in its frame.
- (ii) Assumptions made in the model should be as small as possible.
- (iii) Variables used in the model must be less in number ensuring that it is simple and coherent.
- (iv) It should be open to parametric type of treatment.
- (v) It should not take much time in its construction for any problem.

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### Advantages of a Model

There are certain significant advantages gained when using a model. These are:

- (i) Problems under consideration become controllable through a model.
- (ii) It provides a logical and systematic approach to the problem.
- (iii) It provides the limitations and scope of an activity.
- (iv) It helps in finding useful tools that eliminate duplication of methods applied to solve problems.
- (v) It helps in finding solutions for research and improvements in a system.
- (vi) It provides an economic description and explanation of either the operation, or the systems they represent.

### Classification of Models

The classification of models is a subjective problem. They may be distinguished as follows:

- (i) Models by degree of abstraction
- (ii) Models by function
- (iii) Models by structure
- (iv) Models by nature of an environment
- (v) Models by the extent of generality

### Models by Function

These models consist of (i) Descriptive models, (ii) Predictive models, and (iii) Normative or optimization models.

**Descriptive and Predictive models.** These models describe and predict facts and relationships among the various activities of the problem. These models do not have an objective function as a part of the model to evaluate decision alternatives. In this model, it is possible to get information as to how one or more factors change as a result of changes in other factors.

**Normative or Optimization Models.** They are prescriptive in nature and develop objective decision-rule for optimum solutions.

### Models by Structure

These models are represented by (i) Iconic or Physical models, (ii) Analog models, and (iii) Mathematic or Symbolic models.

**Iconic or Physical Models.** These are pictorial representations of real systems and have the appearance of the real thing. An iconic model is said to be scaled down or scaled up according to the dimensions of the model which may be smaller or greater than that of the real item, e.g., city maps, blue prints of houses, globe and so on. These models are easy to observe and describe but are difficult to manipulate and are not very useful for the purposes of prediction.

**Analog Models.** These are abstract than the iconic ones. There is no look alike correspondence between these models and real life items. The models in

which one set of properties is used to represent another set of properties are called analog models. After the problem is solved, the solution is reinterpreted in terms of the original system. These models are less specific, less concrete but easier to manipulate than iconic models.

**Mathematic or Symbolic Models.** These are most abstract in nature in comparison to others. They employ a set of mathematical symbols to represent the components of the real system. These variables are related together by means of mathematical equations to describe the behaviour of the system. The solution of the problem is then obtained by applying well developed mathematical techniques to the model. The symbolic model is usually the easiest to manipulate experimentally and it is the most general and abstract. Its function is more explanatory than descriptive.

### Models by Nature of an Environment

These models can be classified into (i) Deterministic models, and (ii) Probabilistic or Stochastic models.

**Deterministic Models.** In these models, all parameters and functional relationships are assumed to be known with certainty when the decision is to be made. Linear programming and break even models are the examples of deterministic models.

**Probabilistic or Stochastic Models.** These models are those in which atleast one parameter or decision variable is a random variable. These models reflect to some extent the complexity of the real world and the uncertainty surrounding it.

### Models by the Extent of Generality

These models can be categorized into (i) Specific models, and (ii) General models.

**Specific models.** When a model presents a system at some specific time, it is known as a *specific model*. In these models, if the time factor is not considered them they are termed as *static models*. An inventory problem of determining economic order quantity for the next period assuming that the demand in the planning period would remain same as that of today is an example of static model. *Dynamic programming* may be considered as an example of *dynamic model*.

**General Models.** *Simulation* and *heuristic models* fall under the category of general models. These models are used to explore alternative strategies which have been overlooked previously.

### 9.4.2 Limitations of Operations Research

OR though is a great aid to management as outlined above but still it cannot be a substitute for decision-making. The choice of a criterion as to what is actually best for a business enterprise is still that of an executive who has to fall back upon his

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experience and judgement. This is so because of the several limitations of OR. Important limitations are given below:

1. *The inherent limitations concerning mathematical expressions.* OR involves the use of mathematical models, equations and similar other mathematical expressions. Assumptions are always incorporated in the derivation of an equation or model and such an equation or model may be correctly used for the solution of the business problems when the underlying assumptions and variables in the model are present in the concerned problem. If this caution is not given due care then there always remains the possibility of wrong application of OR techniques. Quite often the operations researchers have been accused of having many solutions without being able to find problems that fit.
2. *High costs are involved in the use of OR techniques.* OR techniques usually prove very expensive. Services of specialized persons are invariably called for and along with it the use of computer and its maintenance is also considered while using OR techniques. Hence, only big concerns can think of using such techniques. Even in big business organizations we can expect that OR techniques will continue to be of limited use simply because they are not in many cases worth their cost. As opposed to this a typical manager, exercising intuition and judgement, may be able to make a decision very inexpensively. Thus, the use of OR is a costlier affair and this constitutes an important limitation of OR.
3. *OR does not take into consideration the intangible factors, i.e., non-measurable human factors.* OR makes no allowance for intangible factors such as skill, attitude, vigour of the management in taking decisions but in many instances success or failure hinges upon the consideration of such non-measurable intangible factors. There cannot be any magic formula for getting an answer to management problems but it depends upon proper managerial attitudes and policies.
4. *OR is only a tool of analysis and not the complete decision-making process.* It should always be kept in mind that OR alone cannot make the final decision. It is just a tool and simply suggests best alternatives. In the final analysis many business decisions will involve human element. Thus, OR is at best a supplement to rather than a substitute for management; subjective judgement is likely to remain a principal approach to decision-making.
5. *Other limitations.* Among other limitations of OR, the following deserve mention:
  - (i) *Bias.* The operational researchers must be unbiased. An attempt to shoehorn results into a confirmation of management's prior preferences can greatly increase the likelihood of failure.



- (ii) *Inadequate objective functions.* The use of a single objective function is often an insufficient basis for decisions. Laws, regulations, public relations, market strategies, etc., may all serve to overrule a choice arrived at in this way.
- (iii) *Internal resistance.* The implementation of an optimal decision may also confront internal obstacles such as trade unions or individual managers with strong preferences for other ways of doing the job.
- (iv) *Competence.* Competent OR analysis calls for the careful specification of alternatives, a full comprehension of the underlying mathematical relationships and a huge mass of data. Formulation of an industrial problem to an OR set programme is quite often a difficult task.
- (v) *Reliability of the prepared solution.* At times a non-linear relationship is changed to linear for fitting the problem to the LP pattern. This may disturb the solution.

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### Check Your Progress

1. How was the concept of operations research started?
2. State one definition of OR.
3. Mention the essential characteristics of operations research.
4. Apart from defence, mention other areas where OR is applied.
5. What are major phases in the application of OR?
6. When was the subject of operational research developed?

## 9.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. The concept of Operations Research came into existence in a military context during World War II, when military management wanted to arrive at decisions on optimal utilization of scarce military resources with a new approach to the systematic and scientific study of the operations of the system.
2. OR is the scientific knowledge through interdisciplinary team effort for the purpose of determining the best utilization of limited resources.
3. Essential characteristics of operations research are:
  - (i) System orientation
  - (ii) Use of interdisciplinary terms
  - (iii) Application of scientific methods
  - (iv) Uncovering new problems
  - (v) Quantitative solutions
  - (vi) Human factors

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4. Following are the areas where concept of OR is applied:
  - (i) Agriculture
  - (ii) Finance
  - (iii) Industry
  - (iv) Marketing
  - (v) Personnel Management
  - (vi) Production Management
  - (vii) Research and Development
5. Major phases involved in the application of OR are:
  - (i) Formulating the problem.
  - (ii) Constructing a mathematical model
  - (iii) Deriving the solution from the model
  - (iv) Testing the model and its solution (updating the model)
  - (v) Controlling the solution
  - (vi) Implementation
6. The subject of Operational Research (OR) was developed in military context during World War II, pioneered by the British scientists.

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## 9.6 SUMMARY

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- The term operations research (OR) was first coined in 1940 by J.F. McCloskey and F.N. Trefethen.
- OR is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control.
- OR can be considered as being the application of scientific method by interdisciplinary teams to problems involving the control of organized (man-machine) systems to provide solutions which best serve the purpose of the organization as a whole.
- OR emphasizes on the overall approach to the system. This characteristic of OR is often referred as system oriented.
- OR involves scientific and systematic attack of complex problems to arrive at the optimum solution.
- Linear programming technique is used in finding a solution for optimizing a given objective, such as profit maximization or cost minimization under certain constraints.
- In OR, waiting line or queuing theory deals with mathematical study of queues which are formed whenever the current demand for service exceeds the current capacity to provide that service.
- In OR, decision theory concerns with making sound decisions under conditions of certainty, risk and uncertainty.
- OR techniques are being used in production, procurement, marketing, finance and other allied fields. Through OR, management can know the reactions of the integrated business systems. The Integrated Production Models technique is used to minimize cost with respect to work force, production and inventory.

- OR provides the business executives such an understanding of the business operations which gives them new insights and capability to determine better solutions for several decision-making problems with great speed, competence and confidence.
- OR study generally involves three phases viz., the judgement phase, the research phase and the action phase.
- The procedure for an OR study generally involves some major steps viz., formulating the problem, constructing the mathematical model to represent the system under study, deriving a solution from the model, testing the model and the solution so derived, establishing controls over the solution and putting the solution to work implementation.
- OR is generally concerned with problems that are tactical rather than strategic in nature.
- A model in OR is a simplified representation of an operation or a process in which only the basic aspects or the most important features of a typical problem under investigation are considered.
- Models by function consist of descriptive models, predictive models and normative models.
- Models by structure are represented by iconic or physical models, analog models and mathematic or symbolic models.
- Models by nature of an environment can be classified into deterministic models and probabilistic models.
- Models by the extent of generality can be categorized into specific models and general models.

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### 9.7 KEY WORDS

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- **Operations research:** The application of scientific knowledge through interdisciplinary team effort for the purpose of determining the best utilization of limited resources.
- **Model:** A simplified representation of an operation or a process that considers only the basic aspects or the most important features of a typical problem under investigation with an objective to identify significant factors and their interrelationships
- **Descriptive model:** A type of model that describes facts and relationships among the various activities of the problem. This model collects information on factors that change as a result of changes in other factors
- **Normative or optimization models:** These models are prescriptive in nature and develop objective decision-rule for optimum solutions

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## 9.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

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#### Short Answer Questions

1. Where did the concept of operations research originate?
2. Name the fields where operations research can be used?
3. What do you mean by nature of operations research?
4. Which Indian companies use operations research?
5. What is a model in operations research?
6. Write any one limitation of operations research.

#### Long Answer Questions

1. Explain the meaning and origin of operations research with the help of definitions and examples.
2. Discuss about the nature of operations research with the help of examples.
3. Write an essay on the development of operations research.
4. Explain the various phases of operations research study.
5. Describe operations research approach with the help of a flow chart.
6. What should be the characteristics of a good operations research model? Explain.
7. Explain the limitations of operations research with the help of suitable examples.

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## 9.9 FURTHER READING

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Taha, H.A. 2006. *Operations Research: An Introduction*, 8th edition. New Delhi: Prentice-Hall of India.

Jensen, Paul A., and Jonathan F. Bard. 2003. *Operations Research Models and Methods*. New York: John Wiley & Sons.

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# UNIT 10 SEQUENCING/ SCHEDULING METHODS

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## NOTES

### Structure

- 10.0 Introduction
- 10.1 Objectives
- 10.2 Terminology and Notations: Concepts
  - 10.2.1 Principal Assumptions for Scheduling Models
  - 10.2.2 Priorities, Processing and Mass Production
- 10.3 Answers to Check Your Progress Questions
- 10.4 Summary
- 10.5 Key Words
- 10.6 Self Assessment Questions and Exercises
- 10.7 Further Reading

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## 10.0 INTRODUCTION

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In this unit, we will determine an appropriate order (sequence) for a series of jobs to be done on a finite number of service facilities in some pre-assigned order, so as to optimize the total cost (time) involved.

Sequencing gives us an idea of the order in which things happen or come in event. Suppose there are  $n$  jobs (1, 2 ...  $n$ ), each of which has to be processed one at a time at  $m$  machines ( $A, B, C \dots$ ). The order of processing each job through each machine is given. The problem is to find a sequence among  $(n!)^m$  number of all possible sequences for processing the jobs so that the total elapsed time for all the jobs will be minimum.

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## 10.1 OBJECTIVES

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After going through this unit, you will be able to:

- Discuss the concepts and terminology of notations
- Analyse the principal assumptions for scheduling models
- Explain the problems concerning priorities, processing and mass production

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## 10.2 TERMINOLOGY AND NOTATIONS: CONCEPTS

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The following are the terminologies and notations used in this unit.

**Number of machines** It means the service facilities through which a job must pass before it is completed.

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**Processing order** It refers to the order in which various machines are required for completing the job.

**Processing time** It means the time required by each job to complete a prescribed procedure on each machine.

**Idle time on a machine** This is the time for which a machine remains idle during the total elapsed time. During the time, the machine awaits completion of manual work. The notation  $x_{ij}$  is used to denote the idle time of a machine  $j$  between the end of the  $(i - 1)$ th job and the start of the  $i$ th job.

**Total elapsed time** This is the time between starting the first job and completing the last job, which also includes the idle time, if it occurs.

**No passing rule** It means, passing is not allowed, i.e., maintaining the same order of jobs over each machine. If each of  $N$ -jobs is to be processed through 2 machines  $M_1$  and  $M_2$  in the order  $M_1 M_2$ , then this rule will mean that each job will go to machine  $M_1$  first and then to  $M_2$ . If a job is finished on  $M_1$ , it goes directly to machine  $M_2$  if it is free, otherwise it starts a waiting line or joins the end of the waiting line, if one already exists. Jobs that form a waiting line are processed on machine  $M_2$  when it becomes free.

**10.2.1 Principal Assumptions for Scheduling Models**

- (i) No machine can process more than one operation at a time.
- (ii) Each operation once started must be performed till completion.
- (iii) Each operation must be completed before starting any other operation.
- (iv) Time intervals for processing are independent of the order in which operations are performed.
- (v) There is only one machine of each type.
- (vi) A job is processed as soon as possible, subject to the ordering requirements.
- (vii) All jobs are known and are ready for processing, before the period under consideration begins.
- (viii) The time required to transfer the jobs between machines is negligible.

**10.2.2 Priorities, Processing and Mass Production**

**Type I: Problems With  $N$  Jobs Through Two Machines**

The algorithm, which is used to optimize the total elapsed time for processing  $n$  jobs through two machines is called ‘Johnson’s algorithm’ and has the following steps.

Consider  $n$  jobs (1, 2, 3 ... $n$ ) processing on two machines  $A$  and  $B$  in the order  $AB$ . The processing periods (time) are  $A_1 A_2 \dots A_n$  and  $B_1 B_2 \dots B_n$  as given in the following table.

<i>Machine/Job</i>	<i>1</i>	<i>2</i>	<i>3</i>	<i>...</i>	<i>n</i>
<i>A</i>	$A_1$	$A_2$	$A_3$	$\dots$	$A_n$
<i>B</i>	$B_1$	$B_2$	$B_3$	$\dots$	$B_n$

The problem is to sequence the jobs so as to minimize the total elapsed time.

The solution procedure adopted by Johnson is given below.

**Step 1** Select the least processing time occurring in the list  $A_1 A_2 \dots A_n$  and  $B_1 B_2 \dots B_n$ . Let this minimum processing time occurs for a job  $K$ .

**Step 2** If the shortest processing is for machine  $A$ , process the  $K^{\text{th}}$  job first and place it in the beginning of the sequence. If it is for machine  $B$ , process the  $K^{\text{th}}$  job last and place it at the end of the sequence.

**Step 3** When there is a tie in selecting the minimum processing time, then there may be three solutions.

- (i) If the equal minimum values occur only for machine  $A$ , select the job with larger processing time in  $B$  to be placed first in the job sequence.
- (ii) If the equal minimum values occur only for machine  $B$ , select the job with larger processing time in  $A$  to be placed last in the job sequence.
- (iii) If there are equal minimum values, one for each machine, then place the job in machine  $A$  first and the one in machine  $B$  last.

**Step 4** Delete the jobs already sequenced. If all the jobs have been sequenced, go to the next step. Otherwise, repeat steps 1 to 3.

**Step 5** In this step, determine the overall or total elapsed time and also the idle time on machines  $A$  and  $B$  as follows.

Total elapsed time = The time between starting the first job in the optimal sequence on machine  $A$  and completing the last job in the optimal sequence on machine  $B$ .

Idle time on  $A$  = (Time when the last job in the optimal sequence is completed on machine  $B$ ) – (Time when the last job in the optimal sequence is completed on machine  $A$ )

Idle time on  $B$  = [when the first job in the optimal sequence starts on machine  $B$  +  $\sum_{K=2}^n$  [time  $k^{\text{th}}$  job starts on machine  $B$  – time  $(K - 1)$ th job finished on machine  $B$ ]

**Example 10.1:** There are five jobs, each of which must go through the two machines  $A$  and  $B$  in the order  $AB$ . Processing times are given below.

Job	1	2	3	4	5
Machine $A$	5	1	9	3	10
Machine $B$	2	6	7	8	4

Determine a sequence for the five jobs that will minimize the total elapsed time.

**Solution:** The shortest processing time in the given problem is 1 on machine  $A$ . So perform job 2 in the beginning, as shown below.

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The reduced list of processing time becomes

<i>Job</i>	<i>1</i>	<i>3</i>	<i>4</i>	<i>5</i>
Machine <i>A</i>	5	9	3	10
Machine <i>B</i>	2	7	8	4

Again the shortest processing time in the reduced list is 2 for job 1 on machine *B*.  
So place job 1 as the last.

2	1
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Continuing in the same manner the next reduced list is obtained as

<i>Job</i>	<i>3</i>	<i>4</i>	<i>5</i>
Machine <i>A</i>	9	3	10
Machine <i>B</i>	7	8	4

Leading to the sequence

2	4	1
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and the list

<i>Job</i>	<i>3</i>	<i>5</i>
Machine <i>A</i>	9	10
Machine <i>B</i>	7	4

gives rise to the sequence

2	4	5	1
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Finally, the optimal sequence *n* is obtained as

2	4	3	5	1
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Flow of jobs through machines *A* and *B* using the optimal sequence is,

$$2 \rightarrow 4 \rightarrow 3 \rightarrow 5 \rightarrow 1.$$

Computation of the total elapsed time and the machine's idle time in hours.

<i>Job</i>	<i>Machine A</i>		<i>Machine B</i>		<i>Idle time</i>	
	<i>In</i>	<i>Out</i>	<i>In</i>	<i>Out</i>	<i>A</i>	<i>B</i>
2	0	1	1	7	0	1
4	1	4	7	15	0	0
3	4	13	15	22	0	0
5	13	23	23	27	0	1
1	23	28	28	30	30 - 28	1
					= 2	3



From the above table we find that the total elapsed time is 30 hours and the idle time on machine *A* is 2 hours and on machine *B* is 3 hours.

**Example 10.2:** Find the sequence that minimizes the total elapsed time (in hours) required to complete the following tasks on two machines.

<i>Task</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>	<i>I</i>
Machine <i>I</i>	2	5	4	9	6	8	7	5	4
Machine <i>II</i>	6	8	7	4	3	9	3	8	11

**Solution:** The shortest processing time is 2 hours on machine *I* for job *A*. Hence, process this job first.

*A*

Deleting these jobs, we get the reduced list of processing time.

<i>Task</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>	<i>I</i>
Machine <i>I</i>	5	4	9	6	8	7	5	4
Machine <i>II</i>	8	7	4	3	9	3	8	11

The next minimum processing time is same for jobs *E* and *G* on machine *II*. The corresponding processing time on machine *I* for this job is 6 and 7. The longest processing time is 7 hours. So sequence job *G* at the end and *E* next to it.

*A*                      *E*                      *G*

Deleting the jobs that are sequenced, the reduced processing list is,

<i>Job</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>F</i>	<i>H</i>	<i>I</i>
Machine <i>I</i>	5	4	9	8	5	4
Machine <i>II</i>	8	7	4	9	8	11

The minimum processing time is 4 hours for job *C*, *I* and *D*. For job *C* and *I* it is on machine *I* and for job *D* it is on machine *II*. There is a tie in sequencing jobs *C* and *I*. To break this, we consider the corresponding time on machine *II*, the longest time is 11 (eleven) hours. Hence, sequence job *I* in the beginning followed by job *C*. For job *D*, as it is on machine *II*, sequences it last.

*A*      *I*      *C*                      *D*      *E*      *G*

Deleting the jobs that are sequenced, the reduced processing list is,

<i>Job</i>	<i>B</i>	<i>F</i>	<i>H</i>
Machine <i>I</i>	5	8	5
Machine <i>II</i>	8	9	8

## NOTES

The next minimum processing time is 5 hours on machine *I* for jobs *B* and *H*, which is again a tie. To break this, we consider the corresponding longest time on the other machine (*II*) and sequence job *B* or *H* first. Finally, job *F* is sequenced.

**NOTES**

The optimal sequence for this job is,

<i>A</i>	<i>I</i>	<i>C</i>	<i>B</i>	<i>H</i>	<i>F</i>	<i>D</i>	<i>E</i>	<i>G</i>
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The total elapsed time and idle time for both the machines are calculated from the following table.

<i>Job</i>	<i>Machine A</i>		<i>Machine B</i>		<i>Idle time</i>	
	<i>In</i>	<i>Out</i>	<i>In</i>	<i>Out</i>	<i>M<sub>1</sub></i>	<i>M<sub>2</sub></i>
<i>A</i>	0	2	2	8	0	2
<i>I</i>	2	6	8	19	0	0
<i>C</i>	6	10	19	26	0	0
<i>B</i>	10	15	26	34	0	0
<i>H</i>	15	20	34	42	0	0
<i>F</i>	20	28	42	51	0	0
<i>D</i>	28	37	51	55	0	0
<i>E</i>	37	43	55	58	0	0
<i>G</i>	43	50	58	61	61 – 50	0
					11 hours	2 hours

Total elapsed time = 61 hours.

Idle time for Machine *I* = 11 hours; Idle time for Machine *II* = 2 hours.

**Type II: Processing *N* Jobs Through Three Machines *A, B, C***

Consider *n* jobs (1, 2 ... *n*) processing on three machines *A, B, C* in the order *ABC*. The optimal sequence can be obtained by converting the problem into a two-machine problem. From this, we get the optimum sequence using Johnson's algorithm.

The following steps are used to convert the given problem into a two-machine problem.

**Step 1** Find the minimum processing time for the jobs on the first and last machine and the maximum processing time for the second machine.

$$\text{i.e., find } \underset{i}{\text{Min}} (A_i, C_i) \quad i = 1, 2 \dots n$$

$$\text{and } \underset{i}{\text{Max}} (B_i)$$

**Step 2** Check the following inequality

$$\underset{i}{\text{Min}} A_i \geq \underset{i}{\text{Max}} B_i$$

or

$$\underset{i}{\text{Min}} C_i \geq \underset{i}{\text{Max}} B_i$$

**Step 3** If none of the inequalities in step 2 are satisfied, this method cannot be applied.

**Step 4** If at least one of the inequalities in step 2 is satisfied, we define two machines  $G$  and  $H$ , such that the processing time on  $G$  and  $H$  are given by,

$$G_i = A_i + B_i \quad i = 1, 2 \dots n$$

$$H_i = B_i + C_i \quad i = 1, 2 \dots n$$

**Step 5** For the converted machines  $G$  and  $H$ , we obtain the optimum sequence using two-machine algorithm.

**Example 10.3:** A machine operator has to perform three operations, turning, threading and knurling, on a number of different jobs. The time required to perform these operations (in minutes) on each job is known. Determine the order in which the jobs should be processed in order to minimize the total time required to turn out all the jobs. Also find the minimum elapsed time.

Job	1	2	3	4	5	6
Turning	3	12	5	2	9	11
Threading	8	6	4	6	3	1
Knurling	13	14	9	12	8	13

**Solution:** Let us consider the three machines as  $A$ ,  $B$  and  $C$ .

$A$  = Turning;  $B$  = Threading;  $C$  = Knurling

**Step 1**  $\text{Min}_i (A_i, C_i) = (2, 8)$   
 $\text{Max}_i (B_i) = 8$

**Step 2**  $\text{Min}_i A_i = 2 \not\geq \text{Max}_i B_i = 8$   
 $\text{Min}_i C_i = 8 \geq \text{Max}_i B_i$  is satisfied.

We define two machines  $G$  and  $H$

Such that,  $G_i = A_i + B_i$   
 $H_i = B_i + C_i$

Job	1	2	3	4	5	6
$G$	11	18	9	8	12	12
$H$	21	20	13	18	11	14

We adopt Johnson's algorithm steps to get the optimum sequence.

4	3	1	6	2	5
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## NOTES

**NOTES**

To find the min. total elapsed time and idle time for machines *A*, *B* and *C*,

<i>Job</i>	<i>Machine A</i>		<i>Machine B</i>		<i>Machine C</i>		<i>Idle time</i>			
	<i>In</i>	<i>Out</i>	<i>In</i>	<i>Out</i>	<i>In</i>	<i>Out</i>	<i>A</i>	<i>B</i>	<i>C</i>	
4	0	2	2	8	8	20	–	2	8	
3	2	7	8	12	20	29	–	–	–	
1	7	10	12	20	29	42	–	–	–	
6	10	21	21	22	42	55	–	1	–	
2	21	33	33	39	55	69	–	11	–	
5	33	42	42	45	69	77	–	3	–	
$77 - 42(77 - 45) + 17$									–	
								35	49	8

Total elapsed time = 77 minutes

Idle time for machine *A* = 35 minutes; Idle time for machine *B* = 49 minutes;

Idle time for machine *C* = 8 minutes.

**Example 10.4:** We have five jobs, each of which must go through the machines *A*, *B* and *C* in the order *ABC*. Determine the sequence that will minimize the total elapsed time.

<i>Job No.</i>	<i>1</i>	<i>2</i>	<i>3</i>	<i>4</i>	<i>5</i>
Machine <i>A</i>	5	7	6	9	5
Machine <i>B</i>	2	1	4	5	3
Machine <i>C</i>	3	7	5	6	7

**Solution:** The optimum sequence can be obtained by converting the problem into that of two-machines, by using the following steps.

**Step 1** Find  $\text{Min}_i (A_i, C_i) \quad i = 1, 2 \dots 5$   
= (5, 3).

**Step 2**  $\text{Max}_i (B_i) = 5$   
 $\text{Min}_i A_i = 5 = \text{Max}_i B_i = 5$   
 $\therefore \text{Min}_i A_i \geq \text{Max}_i B_i$  is satisfied.

We convert the problem into a two-machine problem by defining two machines *G* and *H*, such that the processing time on *G* and *H* are given by,

$$G_i = A_i + B_i$$

$$i = 1, 2, \dots 5$$

$$H_i = B_i + C_i$$

<i>Job</i>	<i>1</i>	<i>2</i>	<i>3</i>	<i>4</i>	<i>5</i>
<i>G</i>	7	8	10	14	8
<i>H</i>	5	8	9	11	10

We obtain the optimum sequence by using the steps in Johnson's algorithm.

5	2	4	3	1
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To find the total elapsed time and idle time on three machines.

Job	Machine A		Machine B		Machine C		Idle time			
	In	Out	In	Out	In	Out	A	B	C	
5	0	5	5	8	8	15	–	5	8	
2	5	12	12	13	15	22	–	4	–	
4	12	21	21	26	26	32	–	8	4	
3	21	27	27	31	32	37	–	1	–	
1	27	32	32	34	37	40	–	1	–	
40 – 32							40 – 34	–	–	–
= 6 + 19							–	–	–	–
8							25	12	–	–

## NOTES

Total elapsed time = 40 hours.

Idle time for machine A = 8 hours; Idle time for machine B = 25 hours; Idle time for machine C = 12 hours.

### Type III: Problems With $N$ Jobs And $K$ Machines

Consider  $n$  jobs (1, 2 ...  $n$ ) processing through  $k$  machines  $M_1 M_2 \dots M_k$  in the same order. The iterative procedure of obtaining an optimal sequence is as follows.

**Step 1** Find Min.  $M_i$  and Min.  $M_{ik}$  and Max. of each of  $M_{i2}, M_{i3} \dots M_{ik-1}$  for  $i = 1, 2 \dots n$ .

**Step 2** Check whether

$$\min_i M_{i1} \geq \max_j M_{ij}, \text{ for } j = 2, 3 \dots k-1 \text{ or}$$

$$\min_i M_{ik} \geq \max_j M_{ij}, \text{ for } j = 2, 3 \dots k-1.$$

**Step 3** If the inequalities in step 2 are not satisfied, the method fails, otherwise, go to the next step.

**Step 4** In addition to step 2, if  $M_{i2} + M_{i3} + \dots + M_{ik-1} = C$ , where  $C$  is a positive fixed constant for all,  $i = 1, 2 \dots n$

Then determine the optimal sequence for  $n$  jobs, where the two machines are  $M_1$  and  $M_k$  in the order  $M_1 M_k$ , by using the optimum sequence algorithm.

**Step 5** If the condition  $M_{i2} + M_{i3} + \dots + M_{ik-1} \neq C$  for all  $i = 1, 2 \dots n$ , we define two machines  $G$  and  $H$  such that,

$$G_i = M_{i1} + M_{i2} + \dots + M_{ik-1}$$

$$H_i = M_{i2} + M_{i3} + \dots + M_{ik} \quad i = 1, 2, \dots, n.$$

Determine the optimal sequence of performance of all jobs on  $G$  and  $H$  using the optimum sequence algorithm for two machines.

**NOTES**

**Example 10.5:** Four jobs 1, 2, 3 and 4 are to be processed on each of the five machines *A, B, C, D* and *E* in the order *A B C D E*. Find the total minimum elapsed time if no passing of jobs is permitted. Also find the idle time for each machine.

Machines	Jobs			
	1	2	3	4
<i>A</i>	7	6	5	8
<i>B</i>	5	6	4	3
<i>C</i>	2	4	5	3
<i>D</i>	3	5	6	2
<i>E</i>	9	10	8	6

**Solution** Since the problem is to be sequenced on five machines, we convert the problem into a two-machine problem by adopting the following steps.

**Step 1** Find  $\text{Min}_i(A_i, E_i) = (5, 6)$

$$i = 1, 2, 3, 4$$

$$\text{Max}(B_i, C_i, D_i) = (6, 5, 6)$$

**Step 2** The inequality

$$\text{Min}_i E_i = 6 \geq \text{Max}_i (B_i, C_i, D_i)$$

is satisfied. Therefore, we can convert the problem into a two-machine problem.

**Step 3** Since  $B_i + C_i + D_i \neq C$ , where  $C$  is a fixed constant, we define two machines  $G$  and  $H$  such that,

$$G_i = A_i + B_i + C_i + D_i$$

$$H_i = B_i + C_i + D_i + E_i \quad i = 1, 2, 3, 4.$$

Job	1	2	3	4
<i>G</i>	17	21	20	16
<i>H</i>	19	25	23	14

1	3	2	4
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Job	Machine A		Machine B		Machine C		Machine D		Machine E	
	In	Out	In	Out	In	Out	In	Out	In	Out
1	0	7	7	12	12	14	14	17	17	26
3	7	12	12	16	14	19	19	25	26	34
2	12	18	18	24	24	28	28	33	34	44
4	18	26	26	29	29	32	33	35	44	50

<i>Idle time</i>				
<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>
–	7	12	14	17
–	–	–	2	–
–	2	5	3	–
–	2	1	–	–
50 – 26	50 – 29 = 21	50 – 32 = 18	50 – 35 = 15	–
24	32	36	34	17

Total elapsed time = 50 hours.

Idle time for machine *A* = 24 hours; Idle time for machine *B* = 32 hours; Idle time for machine *C* = 36 hours; Idle time for machine *D* = 34 hours; Idle time for machine *E* = 17 hours.

#### **Type IV: Problems with 2 jobs through *K* Machines**

Consider two jobs, each of which is to be processed on *K* machines  $M_1, M_2, \dots, M_K$  in two different orders. The ordering of each of the two jobs through *K* machines is known in advance. Such ordering may not be the same for both the jobs. The exact or expected processing times on all the given machines are known.

Each machine can perform only one job at a time. The objective is to determine the optimal sequence of processing the jobs so as to minimize the total elapsed time.

The optimal sequence in this case can be obtained by making use of the graph.

The procedure is given in the following steps.

- Step 1** First draw a set of axes, where the horizontal axis represents processing time on job 1 and the vertical axis represents processing time on job 2.
- Step 2** Mark the processing time for job 1 and job 2 on the horizontal and vertical lines respectively, according to the given order of machines.
- Step 3** Construct various blocks starting from the origin (starting point), by pairing the same machines until the end point.
- Step 4** Draw the line starting from the origin to the end point by moving horizontally, vertically and diagonally along a line which makes an angle of  $45^\circ$  with the horizontal line (base). The horizontal segment of this line indicates that the first job is under process while second job is idle. Similarly, the vertical line indicates that the second job is under process while first job is idle. The diagonal segment of the line shows that the jobs are under process simultaneously.
- Step 5** An optimum path is the one that minimizes the idle time for both the jobs. Thus, we must choose the path on which diagonal movement is maximum.
- Step 6** The total elapsed time is obtained by adding the idle time for either job to the processing time for that job.

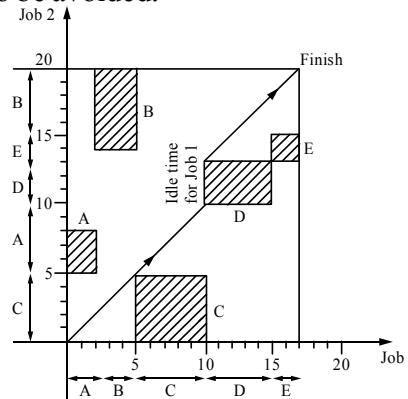
#### **NOTES**

**Example 10.6:** Use graphical method to minimize the time needed to process the following jobs on the machines shown below, i.e., for each machine find the job that should be done first. Also calculate the total time needed to complete both the jobs.

**NOTES**

Job 1	Sequence of machine	A	B	C	D	E
	Time	2	3	4	6	2
Job 2	Sequence of machine	C	A	D	E	B
	Time	4	5	3	2	6

**Solution:** The given information is shown in the figure. The shaded blocks represent the overlaps that are to be avoided.



**Fig. 10.1**

An optimal path is one that minimizes the idle time for job 1 (horizontal movement). Similarly, an optimal path is one that minimizes the idle time for job 2 (vertical movement).

For the elapsed time, we add the idle time for either of the job to the processing time for that job.

In this problem, the idle time for the chosen path is seen to be 3 hours for job 1 and zero for job 2.

Thus, the total elapsed time = 17 + 3 = 20 hours.

**Check Your Progress**

1. What is idle time?
2. What is total elapsed time?
3. State Johnson's algorithm.

**10.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS**

1. Idle time is the time for which a machine remains idle during the total elapsed time. During the time, the machine awaits completion of manual work. The



notation  $x_{ij}$  is used to denote the idle time of a machine  $j$  between the end of the  $(i - 1)$ th job and the start of the  $i$ th job.

2. Total elapsed time is the time between starting the first job and completing the last job, which also includes the idle time, if it occurs.
3. The algorithm, which is used to optimize the total elapsed time for processing  $n$  jobs through two machines is called 'Johnson's algorithm'.

## NOTES

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### 10.4 SUMMARY

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- Sequencing gives us an idea of the order in which things happen or come in event. Suppose there are  $n$  jobs (1, 2 ...  $n$ ), each of which has to be processed one at a time at  $m$  machines ( $A, B, C \dots$ ). The order of processing each job through each machine is given.
- The problem is to find a sequence among  $(n!)m$  number of all possible sequences for processing the jobs so that the total elapsed time for all the jobs will be minimum.
- Idle time on a machine is the time for which a machine remains idle during the total elapsed time. During the time, the machine awaits completion of manual work. The notation  $x_{ij}$  is used to denote the idle time of a machine  $j$  between the end of the  $(i - 1)$ th job and the start of the  $i$ th job.
- Total elapsed time is the time between starting the first job and completing the last job, which also includes the idle time, if it occurs.
- No passing rule means, passing is not allowed, i.e., maintaining the same order of jobs over each machine. If each of  $N$ -jobs is to be processed through 2 machines  $M1$  and  $M2$  in the order  $M1 M2$ , then this rule will mean that each job will go to machine  $M1$  first and then to  $M2$ . If a job is finished on  $M1$ , it goes directly to machine  $M2$  if it is free, otherwise it starts a waiting line or joins the end of the waiting line, if one already exists. Jobs that form a waiting line are processed on machine  $M2$  when it becomes free.
- The algorithm, which is used to optimize the total elapsed time for processing  $n$  jobs through two machines is called 'Johnson's algorithm'.

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### 10.5 KEY WORDS

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- **Total elapsed time:** It is the time between starting the first job and completing the last job, which also includes the idle time, if it occurs.
- **Algorithm:** It is used to optimize the total elapsed time for processing  $n$  jobs through two machines is called 'Johnson's algorithm'.

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## 10.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

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### NOTES

#### Short Answer Questions

1. What is no passing rule in a sequencing algorithm?
2. What is sequencing problem?

#### Long Answer Questions

1. Explain the principle assumptions made while dealing with a sequencing problem.
2. Describe the method of processing  $n$  jobs through two machines.
3. Explain the method of processing  $m$  jobs through three machines  $A$ ,  $B$  and  $C$  in the order  $ABC$ .
4. Explain how to process  $n$  jobs through  $m$  machines.

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## 10.7 FURTHER READING

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Taha, H.A. 2006. *Operations Research: An Introduction*, 8th edition. New Delhi: Prentice-Hall of India.

Jensen, Paul A., and Jonathan F. Bard. 2003. *Operations Research Models and Methods*. New York: John Wiley & Sons.

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# UNIT 11 SIMULATION TECHNIQUES

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## NOTES

### Structure

- 11.0 Introduction
- 11.1 Objectives
- 11.2 Introduction to Simulation as an Aid to Decision Making
  - 11.2.1 Types of Simulation
  - 11.2.2 Random Numbers and Variable
- 11.3 Monte Carlo Simulation: Application of Simulation Models
  - 11.3.1 Monte Carlo Simulation and Inventory Control
  - 11.3.2 Monte Carlo Simulation and Production Line Model
  - 11.3.3 Importance and Uses of Simulation Technique: Advantages and Disadvantages
- 11.4 Cash Management, Project Timing and Product Limitations
- 11.5 Answers to Check Your Progress Questions
- 11.6 Summary
- 11.7 Key Words
- 11.8 Self Assessment Questions and Exercises
- 11.9 Further Reading

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## 11.0 INTRODUCTION

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In this unit, you will learn about simulation. It is a representation of reality through the use of a model or any other device which will react like a real one under a given set of conditions. Simulation can be analog or digital. Digital simulation is also known as computer simulation. You will understand the importance and applications of simulation techniques. The Monte Carlo simulation model used by modern management uses random number tables to reproduce on paper the operation of any given system under its own working conditions. You will also know about the three models of Monte Carlo simulation that are used in three different contexts. These are queuing theory, inventory control and production line.

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## 11.1 OBJECTIVES

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After going through this unit, you will be able to:

- Understand the basic concepts of simulation
- Analyse the applications and importance of simulation
- Discuss the three context models of Monte Carlo simulation

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## 11.2 INTRODUCTION TO SIMULATION AS AN AID TO DECISION MAKING

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### NOTES

**Simulation** is the representation of reality through the use of a model or a device which will react in the same manner as reality under a given set of conditions.

Simulation is also defined as the use of a system model that has the designed characteristics of reality in order to produce the essence of an actual operation.

### 11.2.1 Types of Simulation

Simulation is mainly of two types:

- (i) Analog (environmental) simulation
- (ii) Computer (system) simulation

Some examples of simulation models are as follows:

- (i) Testing an aircraft model in a wind tunnel
- (ii) Children's cycling park with various signals and crossing—to model a traffic system
- (iii) Planetarium

To determine the behaviour of a real system in true environments a number of experiments are performed on simulated models either in the laboratories or in the computer itself.

### 11.2.2 Random Numbers and Variable

The random variable is a real valued function defined over a sample space associated with the outcome of a conceptual chance experiment. Random variables are classified according to their probability density function.

**Random number:** It refers to a uniform random variable or a numerical value assigned to a random variable following uniform probability density function. In other words, it is a number in a sequence of numbers whose probability of occurrence is the same as that of any other number in that sequence.

**Pseudorandom numbers:** Random numbers are called pseudorandom numbers when they are generated by some deterministic process but have already qualified the predetermined statistical test for randomness.

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## 11.3 MONTE CARLO SIMULATION: APPLICATION OF SIMULATION MODELS

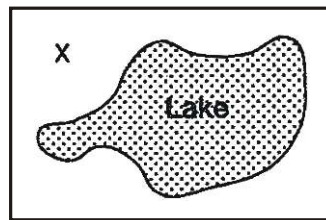
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Various simulation models, based on the principle of similitude (such as the model of aeroplanes initiating flight conditions in a wind tunnel) have been in use for a long time. But Monte Carlo simulation is a recent operations research innovation. The novelty lies in making use of pure chance to contact a simulated version of the

process under analysis in exactly the same way as pure chance operates the original system under working conditions. Only models under uncertainty can be evaluated through Monte Carlo technique. **Monte Carlo technique** can be defined as a technique of simulation in which a series of random numbers are used to create statistical distribution function.

*'Monte Carlo' is the code name given by Von Neumann and S.M. Ulam to the technique of solving problems though it is too expensive for experimental solutions and too complicated for analytical treatment.* Originally, the concept referred to a situation in which a difficult but determinate problem is solved by resort to a chance process. A simple example of this is as follows:

Suppose the problem before us is to find the surface area of an irregularly shaped lake enclosed in a rectangle  $X$  as shown in the given figure:



Instead of finding the area of this lake by geometrical methods we apply the chance process for the purpose. A large number of pebbles are projected by means of a catapult whose elastic is stretched to randomly varying lengths at angles which too vary equally randomly. The area of the lake then is simply,

$$\text{Area of the lake} = X \left( \frac{\text{Number of pebbles falling in water}}{\text{Total number of pebbles projected}} \right)$$

Here,  $X$  means the area of the rectangle enclosing the lake. The underlying rationale is that the probability that a pebble will fall in the water is equal to the ratio of the area of the lake and its enclosing rectangle.

The Monte Carlo simulation technique can as well be used to solve probabilistic problems. Suppose, we are to evaluate the probability  $P$  that a tank will be knocked out by either a first or second shot from an antitank gun assumed to possess a constant kill probability of  $1/2$ . The probability analysis will say that the chance of tank being knocked out by either a first or second shot from an anti-tank gun is  $1/2 + 1/2(1 - 1/2) = 3/4$ . But we can also work out this probability by simulating each round of the anti-tank gun by the flip of a coin through Monte Carlo simulation technique.

Since the probability of a 'head' is the same as that of a kill, we may call it a hit when the coin turns up a head and otherwise a miss. If we flip the coin a large number of times, the value of  $P$  may be calculated by merely counting the number of times the head turns up at least in two successive throws and then dividing this number by the total pairs of throws of the coin. Monte Carlo method in this simple

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case is indeed a poor substitute for the theoretical probability analysis. But many real-life systems are so complicated that even the well defined probability analysis very often fails; but such situations can be handled by Monte Carlo simulation. The Monte Carlo technique provides the simplest possible solutions for queuing problems. Problems of corporate planning, inventory control, capital investment, consumer behaviour and quality control can also be handled through simulation.

Monte Carlo simulation uses random number tables to reproduce on paper the operation of any given system under its own working conditions. This technique is used to solve problems that depend upon probability where formulation of mathematical model is not possible. It involves first, the determining of the probability distribution of the concerned variables and then sampling from this distribution by means of random numbers to obtain data. It may, however, be emphasized here that the probability distributions to be used should closely resemble the real world situation.

One should always remember that simulation is not a perfect substitute but rather an alternative procedure for evaluating a model. Analytical solution produces the optimal answer to a given problem, while Monte Carlo simulation yields a solution which should be very close to the optimal but not necessarily the exact correct solution. The solution of Monte Carlo simulation converges to the optimal solution as the number of simulated trials goes to infinity.

**In context of Queuing Theory:** Monte Carlo simulation which uses random number tables can better be illustrated by considering any concrete operation subject to chance. Let us take the arrival of scooters at a service station. First, observe the actual arrivals of scooters on a number of days, say for five days. Then put this information in the following two ways:

- Group the number of scooters which arrive every hour, say between 7–8, 8–9 a.m. and so on till 4–5 p.m. (assuming the working hours of the service station is from 7 a.m. to 5 p.m. and also assuming that no scooter arrives before 7 a.m. nor any scooter after 5 p.m.). Work out the mean number of scooters which arrive during 7–8 a.m., 8–9 a.m., 9–10 a.m. and so on. Let us suppose, we get the information as given in Table 11.1.

*Table 11.1 Mean Number of Scooters Arriving per Hour*

7–8 a.m.	5.6
8–9 a.m.	5.4
9–10 a.m.	3.4
10–11 a.m.	3.6
11–12 noon	2.0
12–1 p.m.	3.0
1–2 p.m.	4.0
2–3 p.m.	6.0
3–4 p.m.	3.0
4–5 p.m.	4.0

- Obtain the deviation of actual arrivals during a particular hour from the corresponding mean and do it for all the hours from 7 a.m. to 5 p.m. There will thus be ten sets of five deviations (because of the observation on five days) from each of the 10 mean-hourly arrivals; then prepare a frequency distribution of such deviations. Suppose we get the frequency distribution of such deviations as given in Table 11.2.

*Table 11.2 Random Number Allotment*

<i>Deviation from Mean</i>	<i>Frequency</i>	<i>Percentage Frequency</i>	<i>Probability</i>	<i>Random Nos. Allotted</i>
+4.0	2	4	0.04	00–03
+3.5	3	6	0.06	04–09
+3.0	5	10	0.10	10–19
+2.5	4	8	0.08	20–27
+2.0	6	12	0.12	28–39
+1.5	2	4	0.04	40–43
+1.0	3	6	0.06	44–49
+0.5	4	8	0.08	50–57
–0.5	3	6	0.06	58–63
–1.0	2	4	0.04	64–67
–1.5	4	8	0.08	68–75
–2.0	2	4	0.04	76–79
–2.5	3	6	0.06	80–85
–3.0	1	2	0.02	86–87
–3.5	2	4	0.04	88–91
–4.0	4	8	0.08	92–99
Total	50	100	1.00	

In the last column of the above table we have allotted 100 random numbers from 00 to 99, both inclusive according to the percentage frequency of the deviation or the probability distribution of deviations. Thus, deviation to the extent of +4.0 from the mean, having a frequency of 2 out of 50 (and as such 4 out of 100) or a probability of 0.04, has been allotted 4 random numbers, 00 to 03. The next deviation of +3.5 with a probability of 0.06 has the next 6 random numbers, 04 to 09 allotted to it. The same treatment has been done to all the remaining deviations. The last deviation – 4.0 with a probability of 0.08 has been allotted to the last 8 random numbers, from 92 to 99.

The object of doing all this is to derive by simulation the actual number of scooters that may be expected to arrive during any given hour. Suppose we want to know the expected number of scooters arriving on a particular day during the hour 8 to 9 a.m. The table giving the mean number of scooters arrival shows that the mean arrival during this hour is 5.4. If we can ascertain the deviation of the actual arrivals from the mean, we can easily work out the actual number of scooters arrived. To do so, we look at the table of random numbers and select any two-digit number at random. Suppose, the random number so selected is 84 corresponding to which the value of the deviation of actual arrivals from the mean as per the above table is –2.5. In other words the actual arrivals for the hour 8–9 a.m. will be  $(5.4) - (2.5) = 2.9$  (or approximately 3), so that we may say that

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three scooters will arrive between 8–9 a.m. on the day in question. *The underlying rationale of this simulation procedure is that every deviation from its corresponding mean has the same chance of occurring by random number selection as in the actual case because each deviation has as many random numbers allotted to it as its frequency percentage in the general pool of all deviations as stated above.*

We can go a step further and using the random number technique can even simulate and tell the actual arrival time of the scooter coming to the service station during any particular hour. If we are satisfied to note the arrival time correct to within, say 5 minutes, the required numbers of minutes past the hour can take a value only in one of the 12 intervals (05, 6-10, 11-15, ..., 56-60 minutes) into which any hour can be divided. Keeping this in view and the actual observations for all five days under consideration, we can prepare a frequency table showing how many scooters arrive within 5 minutes, how many within 6-10 minutes past the hour, and so on for the remaining intervals into which we choose to split the hour. Let the observed information for 5-day period on this basis be put as shown in Table 11.3.

Random numbers in the last column of Table 11.3 have been allotted in a similar manner as we did in an earlier table. We have already seen that 3 scooters arrive during the hour 8-9 a.m. and now we want to know the actual time of their coming to the service station.

**Table 11.3** *Number of Scooters arriving Within the Number of Minutes Past the Hour*

<i>Deviation from Mean</i>	<i>Frequency</i>	<i>Percentage Frequency</i>	<i>Probability</i>	<i>Random Nos. Allotted</i>
0-5 min.	20	10	0.10	00-09
6-10 min.	30	15	0.15	10-24
11-15 min.	10	5	0.05	25-29
16-20 min.	40	20	0.20	30-49
21-25 min.	16	8	0.08	50-57
26-30 min.	14	7	0.07	58-64
31-35 min.	18	9	0.09	65-73
36-40 min.	12	6	0.06	74-79
41-45 min.	16	8	0.08	80-87
46-50 min.	14	7	0.07	88-94
51-55 min.	6	3	0.03	95-97
56-60 min.	4	2	0.02	98-99
Total	200	100	1.00	

For this purpose, we pick a two-digit random number from the table of random numbers and let us say it is 25. A reference to the above table shows that this



number occurs in the range of 25–29 which belongs to the interval 11–15 minutes and this means that the first of the three scooters arriving between 8–9 a.m. arrives at 15 minutes past 8 a.m. Similarly, picking two more random numbers, viz., 36 and 96, we find from the above table that they are related to intervals 16–20 minutes and 51–55 minutes respectively. Thus, the second scooter arrives at 20 minutes past 8 a.m. and the third scooter arrives at 55 minutes past 8 a.m.

Proceeding in a similar manner we can also make a frequency table showing the number of scooters serviced within intervals of varying magnitudes. Each of these intervals can then be allotted a set of 100 random numbers (i.e., 00 to 99) according to the probability distribution to provide a basis for simulating the pattern of available service.

The above is an example of how arrival as well as service pattern in a queuing process may be derived by Monte Carlo simulation. Remember that there is no regularity either in the arrival of the scooters or in rendering service and because of this there may be times when scooters have to wait for service while at other times the service attendant may remain idle. If in such a case we want to add one more service point to the service station, then certainly we would first like to assess whether the same would be economical or not. Simulation technique can assist us in the matter.

**Example 11.1:** A firm has a single channel service station with the following empirical data available to its management:

- (i) The mean arrival rate is 6.2 minutes.
- (ii) The mean service time is 5.5 minutes.
- (iii) The arrival and service time probability distributions are as follows:

<i>Arrivals</i>	<i>Probability (Minutes)</i>	<i>Service Time</i>	<i>Probability (Minutes)</i>
3-4	0.05	3-4	0.10
4-5	0.20	4-5	0.20
5-6	0.35	5-6	0.40
6-7	0.25	6-7	0.20
7-8	0.10	7-8	0.10
8-9	0.05	8-9	0.00
	1.00		1.00

The queuing process begins at 10.00 a.m. and proceeds for nearly 2 hours. An arrival goes to the service facility immediately if it is empty; otherwise, it will wait in a queue. The queue discipline is, first come first served.

If the attendant's wage is ₹ 8 per hour and the customer's waiting time cost ₹ 9 per hour, would it be an economical proposition to engage second attendant? Answer on the basis of Monte Carlo simulation technique. You may use the figures based upon the simulated period for 2 hours.

**Solution:** From the given probability distributions of arrivals and service times, first we allot the random numbers to the various intervals. This has been done as follows:

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<i>Arrivals (Minutes)</i>	<i>Probability</i>	<i>Random Nos. Allotted</i>	<i>Service Time (Minutes)</i>	<i>Probability</i>	<i>Random Nos. Allotted</i>
3-4	0.05	00-04	3-4	0.10	00-09
4-5	0.20	05-24	4.5	0.20	10-29
5-6	0.35	25-59	5-6	0.40	30-69
6-7	0.25	60-84	6-7	0.20	70-89
7-8	0.10	85-94	7-8	0.10	90-99
8-9	0.05	95-99	8-9	0.00	

These allotted random numbers now become the basis for generating the arrival and service times in conjunction with a table of random numbers. A simulation worksheet as shown on the next page has been developed in the following manner:

As the given mean arrival rate is 6.2 minutes, it means that in an hour approximately 10 units arrive and as such, in a simulation period for 2 hours about 20 units are expected to arrive. Hence, the number of arrivals for our simulation exercise is 20.

A table of random numbers (given in Table 11.4) is used for developing the simulation worksheet. The first random number for arrival time is 44. This number lies between 25-59 and this indicates a simulated arrival time of 5 minutes. All simulated arrival and service times have been worked out in a similar fashion.

The next step is to list the arrival time in the appropriate column. The first arrival comes in 5 minutes after the starting time. It means that the attendant waited for 5 minutes. This has been shown under the column 'Waiting Time: Attendant'. The simulated service time for the first arrival is 5 minutes which results in the service being completed by 10.10 a.m. The next arrival is at 10.11 a.m. which indicates that no one has waited in queue, but the attendant has waited for 1 minute from 10.10 a.m. to 10.11 a.m. The service time ends at 10.18 a.m. But the third arrival is at 10.17 a.m. and the service of the second arrival continues upto 10.18 a.m., hence the third arrival has to wait in the queue. This is shown in the column 'Waiting Time: Customer' of the simulation worksheet. One customer waiting in queue is shown in the column—'Length of the Line.' The same procedure has been followed throughout the preparation of the simulation worksheet.

The following information can be derived from the above stated simulation worksheet.

1. Average length of queue:

$$= \frac{\text{No. of customers in line}}{\text{No. of arrivals}} = \frac{14}{20} = 0.70$$

2. Average waiting time of customer before service:

$$= \frac{\text{Customer waiting time}}{\text{No. of arrivals}} = \frac{41}{20} = 2.05 \text{ minutes}$$

Table 11.4 Simulation Worksheet for Arrival Time, Service Time and Waiting Time

Random Number	Time Till Next Arrival (Minutes)	Arrival Time (a.m.)	Service Begins (a.m.)	Random Number	Service Time (Minutes)	Service Ends (a.m.)	Waiting Time (Minutes)	Length of Customers (Minutes)	The Line
44	5	10.05	10.05	50	5	10.10	5	-	-
84	6	10.11	10.11	95	7	10.18	1	-	1
82	6	10.17	10.18	58	5	10.23	-	1	1
50	5	10.22	10.23	44	5	10.28	-	1	-
83	6	10.28	10.28	77	6	10.34	-	-	1
40	5	10.33	10.34	11	4	10.38	-	1	-
96	8	10.41	10.41	08	3	10.44	3	-	-
88	7	10.48	10.48	38	5	10.53	4	-	1
16	4	10.52	10.53	87	6	10.59	-	1	1
16	4	10.56	10.59	45	5	11.04	-	3	-
97	8	11.04	11.04	09	3	11.07	-	-	-
92	7	11.11	11.11	99	7	11.18	4	-	1
39	5	11.16	11.18	81	6	11.24	-	2	1
33	5	11.21	11.24	97	7	11.31	-	3	1
83	6	11.27	11.31	30	4	11.35	-	4	1
42	5	11.32	11.35	36	5	11.40	-	3	1
16	4	11.36	11.40	75	6	11.46	-	4	1
07	4	11.40	11.46	72	6	11.52	-	6	1
77	6	11.46	11.52	79	6	11.58	-	6	1
66	6	11.52	11.58	83	6	12.04	-	6	-
20	112				107		17	41	13

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## 3. Average service time:

$$= \frac{\text{Total service time}}{\text{No. of arrivals}} = \frac{107}{20} = 5.35 \text{ minutes}$$

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## 4. Time a customer spends in the system:

$$= \text{Average service time} + \text{Average waiting time before service}$$

$$= 5.35 + 2.05 = 7.40 \text{ minutes}$$

Simulation worksheet developed above also states that if one more attendant is added then there is no need for a customer to 'Wait in Queue'. But the cost of having one more attendant in addition to the existing one is to be compared with the cost of one attendant and the customer waiting time. This can be worked out as under:

<i>Two-hour Period</i>	<i>Cost With</i>	
	<i>One Attendant</i>	<i>Two Attendants</i>
<i>Attendants</i>		
Customer waiting time (41 mts × ₹ 9 per hour)	₹ 6.15	nil
Attendant's cost (2 hours × ₹ 8 per hour)	₹ 16	₹ 32
Total	₹ 22.15	₹ 32

If the above analysis is based on simulation for a period of 2 hours and is representative of the actual situation, then it can be concluded that the cost with one attendant is lower than with two attendants. Hence, it would not be an economical proposition to engage an additional attendant.

**11.3.1 Monte Carlo Simulation and Inventory Control**

**Example 11.2:** Suppose that the weekly demand of Electric Motors has the following probability distribution:

<i>Number Demanded</i>	<i>Probability</i>	<i>Random Numbers Assigned</i>
0	0.10	00 to 09
1	0.40	10 to 49
2	0.30	50 to 79
3	0.20	80 to 99
Total	1.00	

The distribution pattern of delivery time was as follows:

<i>Number of Weeks</i>	<i>Probability from Order to Delivery</i>	<i>Random Numbers Assigned</i>
2	0.20	00 to 19
3	0.60	20 to 79
4	0.20	80 to 99
Total	1.00	

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Inventory carrying cost is ₹ 5 per unit per week, order placing cost is ₹ 10 per occurrence and loss in net revenue (sale price less cost of goods) is ₹ 50 per unit from shortage.

Estimate the average weekly cost of the inventory system with a policy of using reorder quantities of 4 and a reorder points of 5 units using the technique of Monte Carlo simulation for 20-weeks period taking 8 units as the opening balance of inventory.

**Solution:** We shall first develop the simulation worksheet, taking into account all the given information.

Simulation worksheet for a period of 20 weeks concerning the inventory system and the related costs with reorder point of 5 units and reorder quantity of 4 units is given below.

<i>Simulated Costs Number</i>	<i>Random Numbers Week Number Tables)</i>		<i>Simulated Weekly (using Random</i>			<i>Simulated Activity Shortage</i>		
	<i>For Demand</i>	<i>For Delivery Time</i>	<i>Demand Units</i>	<i>Receipt Units</i>	<i>Balance Units</i>	<i>Inventory Holding (in ₹)</i>	<i>Ordering (in ₹)</i>	<i>(in ₹)</i>
<i>1</i>	<i>2</i>	<i>3</i>	<i>4</i>	<i>5</i>	<i>6</i>	<i>7</i>	<i>8</i>	<i>9</i>
0					8*			
1	44		1		7	35		
2	84	82	3		4	20	10	
3	50		2		2	10		
4	85		3		(-1)			50
5	40		1		(-1)			50
6	96	88	3	4	1	5	10	
7	16		1					
8	16		1		(-1)			50
9	97		3		(-1)			150
10	92	39	3	4	1	5	10	

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11	33		1					
12	83		3		(-3)			150
13	42	16	1	4	3	15	10	
14	07		0		3	15		
15	77	66	2	4	5	25	10	
16	50		2		3	15		
17	20		1		2	10		
18	50	95	2	4	4	20	10	
19	58		2		2	10		
20	44		1		1	5		
Total			36	20	38	190	60	450
Average			1.8	1	1.9	9.5	3	22.5

The procedure adopted in developing the above work sheet is briefly explained as under.

<i>Successive Week Numbers</i>	<i>Occurrence of Events and their Effects</i>
1	Random number '44' indicates in column of given demand pattern that customers demanded 1 unit leaving an inventory balance of $8 - 1 = 7$ units at a carrying cost of ₹ 5 per unit which equals ₹ 35.
2	Random number '84' indicates a demand for 3 units and leaves an inventory balance of 4 units which is less than the reorder point of 5 units. It costs ₹ 10 to reorder. New order was placed. Random number '82' indicates that it will take 4 weeks to receive the new units ordered.
3	No special event.
4	There was a demand for 3 units but only 2 units were sold because of inventory exhaustion. Under the assumption that customers went elsewhere, one unit not sold represents a loss in net revenue to the extent of ₹ 50.
5	New stock still has not arrived and a further loss of potential sales of 1 unit at a cost of ₹ 50 occurred.
6	The quantity of 4 units ordered in week 2 has arrived, of which 3 units were sold and 1 unit was in balance. Hence, new order was placed.
7	Neither there was any shortage nor there was anything left in inventory balance.
8 and 9	Shortage to the extent of 1 unit and unit 3 respectively remained causing a potential loss of ₹ 50 and ₹ 150 respectively.
10	Ordered quantity in week six arrived, three units were sold and one remained in balance. Fresh order was placed.

11	No shortage and no balance.
12	Shortage and potential loss of ₹ 150.
13	Ordered quantity in week 10 arrived, 1 unit was sold and three remained in balance. Fresh order was placed.
14	No special event.
15	Ordered quantity in week 13 arrived, 2 units sold and 5 units remained in balance. Fresh order was placed.
16 and 17	No special event.
18	Ordered quantity in week 25 arrived, 2 units sold and 4 remained in balance. Fresh order was placed.
19 and 20	No special event.

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Under the assumptions of the simulation procedure it is noted that weekly costs averaged ₹ 9.5 to carry inventory, ₹ 3 to order and ₹ 22.5 from shortages. In all it costs ₹ 35 per week approximately to maintain the inventory system with a policy of using reorder quantities of 4 and a reorder of 5 units.

**11.3.2 Monte Carlo Simulation and Production Line Model**

**Example 11.3:** A certain production process produces on an average seven per cent defective items. Defective items occur randomly. Items are packaged for sale in lots of five. The production manager wants to know what percentage of the lot contains no defectives. You are required to solve the manager's problem using Monte Carlo simulation. You may as well compare your simulated results with those obtainable using analytical methods. If there remains any difference between such results then account for the same. (Simulate at least 10 lots each of five items).

**Solution:** Since on an average the process produces 7% defective items, so 93% of the items produced are good. For this, we can have the following probability distribution:

Item	Probability	Random Nos. Assigned
Defective	07	00 to 06
Good	093	07 to 99
Total	1.00	

Using the random number table develop the simulation worksheet as follows:

**Simulation worksheet for 16 lots taken for finding the percentage of lots with no defectives.**

Lots	Random Numbers					Simulated Items (Five for Each Lot) G=Good; D=Defective					Defective
	Number in Lot										
1	44	84	82	50	85	G	G	G	G	G	0
2	40	96	88	16	16	G	G	G	G	G	0
3	97	92	39	33	83	G	G	G	G	G	0
4	42	16	07	77	66	G	G	G	G	G	0

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5	50	20	50	95	58	G	G	G	G	G	0
6	44	77	11	08	38	G	G	G	G	G	0
7	87	45	09	99	81	G	G	G	G	G	0
8	97	30	36	75	72	G	G	G	G	G	0
9	79	83	07	00	42	G	G	G	D	G	1
10	13	97	16	83	11	G	G	G	G	G	0
11	45	65	34	89	12	G	G	G	G	G	0
12	64	86	46	32	76	G	G	G	G	G	0
13	07	51	25	36	19	G	G	G	G	G	0
14	32	14	31	96	03	G	G	G	G	D	1
15	93	16	62	24	08	G	G	G	G	G	0
16	38	88	74	47	00	G	G	G	G	D	1

On the basis of these simulated sample of 16 lots, we can estimate the percentage of good lots as =  $\frac{13}{16} \times 100 = 81.25$

If we use the analytical method or the well-formulated mathematical method, we get the following result:

$$p = p. (\text{Good item}) = 0.93$$

$$q = p. (\text{Defective item}) = 0.07$$

Each lot is of 5 items which means that  $n = 5$ .

Using the binomial probability function, we have the following:

$$\begin{aligned} p (\text{all five good items}) &= {}^5C_5 (0.93)^5 \cdot (0.07)^0 \\ &= 1 \cdot (0.93)^5 \cdot 1 \\ &= 0.70 \\ &= 70\% \end{aligned}$$

This analytical answer equals the true expected value or the average number of good lots in the long run. But the simulated result gives this percentage as 81.25 which differs from that of 70% obtained by the mathematical method. The difference is on account of the fact that we took only 16 lots for simulation exercise. If we increase this number, then our answer will approximate to that of the answer from analytical method. Thus, simulation gives only the best possible estimates and not the optimal result as given by analytical methods.

### 11.3.3 Importance and Uses of Simulation Technique: Advantages and Disadvantages

Monte Carlo simulation is often used by modern management when it cannot use other techniques. There are many industrial problems which defy mathematical solutions. The reason is that either they are too complicated or that the data cannot be expressed in mathematical terms. In such cases, it is still possible to reach valid conclusions by using the Monte Carlo technique. A considerable help is thus obtained at practically no cost in taking decisions concerning the functioning of a business system. The data and conclusions can be obtained through simulation of an actual operation on the basis of its own past working. It paves the way for



predicting the changes in its behaviour and the result is evaluated from innovations that we want to introduce.

By using a fresh series of random numbers at the appropriate junctures we can also examine the reactions of the simulated model just as if the same alterations had actually been made in the system itself. Monte Carlo simulation, therefore, provides a tool of knowing in advance whether or not the expense to be incurred or the investment to be made in making the changes envisaged. Through this technique you can introduce the innovations on a piece of paper, examine their effects and then may decide to adopt or not to adopt such innovations in the functioning of real system. The usefulness of simulation lies in the fact that it allows us to experiment with a model of the system rather than the actual system; in case we are convinced about the results of our experiments we can put the same into practice. *Thus the effect of the actual decisions are tested in advance through the technique of simulation by resorting to the study of the model representing the real life situation or the system.*

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**Table 11.5** Simulation Worksheet for simulating Sizes to locate the Number of Misfits

Assembly S.N.	Shafts		Rings			
	Random Numbers (from Tippett tables)	Random Normal Deviate (z)	Simulating Size $X = \mu + z(\sigma)$	Random Numbers (from Tippett tables)	Random Normal Deviate (z)	Simulating Size $X = \mu + z(\sigma)$
1	2952	0.82	$0.980 + 0.82(0.01)$ = 0.9882	3992	1.28	$1.0 + 1.28(0.02)$ = 1.0256
2	3170	0.91	$0.980 + 0.91(0.01)$ = 0.9891	4167	1.38	$1.0 + 1.38(0.02)$ = 1.0276
3	7203	-0.59	$0.980 - 0.59(0.01)$ = 0.9741	1300	0.33	$1.0 + 0.33(0.02)$ = 1.0066
4	3408	1.00	$0.980 + 1.0(0.01)$ = 0.9900	3563	1.06	$1.0 + 1.06(0.02)$ = 1.0212
5	0560	0.14	$0.980 + 0.14(0.01)$ = 0.9814	1112	0.28	$1.0 + 0.28(0.02)$ = 1.0056
6	6641	-0.42	$0.980 - 0.42(0.01)$ = 0.9758	9792	-2.04	$1.0 - 2.04(0.02)$ = 0.9592
7	5624	-0.16	$0.980 - 0.16(0.01)$ = 0.9784	9525	-1.67	$1.0 - 1.67(0.02)$ = 0.9666
8	5356	-0.09	$0.980 - 0.09(0.01)$ = 0.9791	2693	0.74	$1.0 + 0.74(0.02)$ = 1.0148
9	2769	0.76	$0.980 + 0.76(0.01)$ = 0.9876	6107	-0.28	$1.0 - 0.28(0.02)$ = 0.9944
10	5246	-0.06	$0.980 - 0.06(0.01)$ = 0.9794	9025	-1.29	$1.0 - 1.29(0.02)$ = 0.9742

The main purpose of simulation in management is to provide feedback which is vital for the learning process. It creates an atmosphere in which managers play a

**NOTES**

dynamic role by enriching their experience through involvement in reckoning with actual conditions through experimentation on paper. The technique permits trying out several alternatives as the entire production for service process can be worked out on paper, without dislocating the system in any way. Thus, Monte Carlo technique transforms the manager from a blindfolded driver of an automobile, reacting to instructions of a fellow passenger to one who can see fairly and clearly, where he is going.

**Areas of Application of Monte Carlo Simulation**

Monte Carlo simulation has been applied to a wide diversity of problems ranging from queuing process, inventory problem, risk analysis concerning a major capital investment such as the introduction of a new product, expansion of capacity and many other problems. *Budgeting* is another area where simulation can be very useful. In fact, the system of flexible budgeting is an exercise in simulation. Simulation can as well be used for preparing the master budget through functional budgets.

Over and above, the greatest contribution of simulation is in the analysis of complex systems, ‘many real-world problems involve systems made up of many components parts that are interrelated. The system may be dynamic and changing over time and may involve probabilistic or uncertain events. Simulation is the only technique for quantitative analysis of such problems.’

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## **11.4 CASH MANAGEMENT, PROJECT TIMING AND PRODUCT LIMITATIONS**

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Monte Carlo simulation methods in finance are specifically used to calculate the value of companies, to estimate investments in projects and to appraise financial derivatives. It models the project schedules for estimating worst-case, best-case and most possible period for each task to determine outcomes for the overall project. Hence, Monte Carlo simulation helps in analysing the project risks. It helps project managers to include uncertainty and risk in their project plans and accordingly define logical expectations on their projects concerning both schedule and budget.

Basically, using the Monte Carlo method a problem can be directly solved by simulating the underlying physical process and then calculating the average result of the process. Simulation is used in inventory control problems to fulfill the order of customers with proper consideration of product demand during lead time. Here both the lead time and the inventory per unit time are random variables. The following examples explain the project timing, product limitations and cash management for a demand.

**Example 11.4:** A sample of 100 arrivals of a customer at a retail sales depot is according to the following distribution.

<i>Time between arrival (min.)</i>	<i>Frequency</i>
0.5	2
1.0	6
1.5	10
2.0	25
2.5	20
3.0	14
3.5	10
4.0	7
4.5	4
5.0	2

**NOTES**

A study of the time required to service customers by adding up the bills, receiving payments and placing packages, yields the following distribution.

<i>Time between arrival (min.)</i>	<i>Frequency</i>
0.5	12
1.0	21
1.5	36
2.0	19
2.5	7
3.0	5

Estimate the average percentage of customer waiting time and average percentage of idle time of the server by simulation for the next 10 arrivals.

**Solution:** Tag-numbers are allocated to the events in the same proportions as indicated by the probabilities.

<i>Arrivals</i>	<i>Frequency</i>	<i>Probability</i>	<i>Cumulative Probability</i>	<i>Tag-Numbers</i>
0.5	2	0.02	0.02	00–01
1.0	6	0.06	0.08	02–07
1.5	10	0.10	0.18	08–17
2.0	25	0.25	0.43	18–42
2.5	20	0.20	0.63	43–62
3.0	14	0.14	0.77	63–76
3.5	10	0.10	0.87	77–86
4.0	7	0.07	0.94	87–93
4.5	4	0.04	0.98	94–97
5.0	2	0.02	1.00	98–99

## NOTES

Service time (min.)	Frequency	Probability	Cumulative Probability	Tag-Numbers
0.5	12	0.12	0.12	00–11
1.0	21	0.21	0.33	12–32
1.5	36	0.36	0.69	33–68
2.0	19	0.19	0.88	69–87
2.5	7	0.07	0.95	88–94
3.0	5	0.05	1.00	95–97

The random numbers are generated and linked to the appropriate events. The first 10 random numbers, simulating arrival, the second 10, simulating service times. The results are incorporated in Table below, on the assumption that the system starts at 0.00 am.

Average waiting time per customer is,  $\frac{4.5}{10} = 0.45$  minutes.

Average waiting time (or) idle time of the servers =  $\frac{7.00}{10} = 0.7$  minutes.

Arrival no.	Random Number	Inter-Arrival Time (min.)	Arrival Time (min.) <i>a</i>	Random no.	Service Time (min.) <i>e</i>	Service Start <i>b</i>	Service end <i>f</i>	Waiting Time of Customer <i>c = d = f - a</i>	Waiting Time of Server <i>c = d = f - a</i>
1	93	4.0	4.0	78	2.0	4	6	–	40
2	22	2.0	6.0	76	2.0	6	8	–	–
3	53	2.5	8.5	58	1.5	8.5	10.0	–	0.5
4	64	3.0	11.5	54	1.5	11.5	13	–	1.5
5	39	2.0	13.5	74	2.0	13.5	15.5	–	0.5
6	07	1.0	14.5	92	2.5	15.5	18	1.0	–
7	10	1.5	16.0	38	1.5	18.0	19.5	2.0	–
8	63	3.0	19.0	70	2.0	19.5	21.5	0.5	–
9	76	3.0	22.0	96	3.0	22.0	25.0	–	0.5
10	35	2.0	24.0	92	2.5	25.0	27.5	1.0	–
Total								4.5	7.0

**Example 11.5:** A tourist car operator finds that during the past few months, the car's use has varied so much that the cost of maintaining the car varied considerably. During the past 200 days the demand for the car fluctuated as below.

Trips per Week	Frequency
0	16
1	24
2	30
3	60
4	40
5	30

Using random numbers, simulate the demand for a 10-week period.

**Solution:** The tag-numbers allotted for the various demand levels are shown in the table below.

<i>Trips/week or Demand/week</i>	<i>Frequency</i>	<i>Probability</i>	<i>Cumulative Probability</i>	<i>Tag-Numbers</i>
0	16	0.08	0.08	00–07
1	24	0.12	0.20	08–19
2	30	0.15	0.35	20–34
3	60	0.30	0.65	35–64
4	40	0.20	0.85	65–84
5	30	0.15	1.00	85–99

## NOTES

<i>Week</i>	<i>Random number</i>	<i>Demand</i>
1	82	4
2	96	5
3	18	1
4	96	5
5	20	2
6	84	4
7	56	3
8	11	1
9	52	3
10	03	0
		Total = 28

The simulated demand for the cars for the next 10 weeks period is given in the table above.

Total demand = 28 cars.

Average demand =  $\frac{28}{10} = 2.8$  cars per week.

### Capital Budgeting Problem

When uncertainty haunts in the estimation of variables in a capital budgeting exercise, simulation technique may be used with respect to few of the variables, taking the other variables at their best estimates.

We know that P, V, F, Q, T, K, I, D and N are the important variables. (P – Price per unit of output, V – Variable cost per unit of output, F – Fixed cost of operation, Q – Quantity of output, T – Tax rate, K – Discount rate or cost of capital, I – Original investment, D – Annual depreciation and N – Number of years of the project's life).

Suppose in a project, P, V, F, Q, N and I are fairly predictable but K and T are playing truant. In such cases, the K and T will be dealt through simulation while others take given values.

Suppose that  $P = ₹ 300/\text{unit}$ ,  $V = ₹ 150/\text{unit}$ ,  $F = ₹ 15,00,000/\text{p.a.}$ ,  $Q = 20,000/\text{p.a.}$ ,  $N = 3$  years and  $I = ₹ 18,00,000$ . Then annual profit before tax =  $[(P-V)Q] - F - D = [(300-150) \times 20000] - 15,00,000 - 6,00,000 = ₹ 9,00,000/\text{p.a.}$

## NOTES

The profit after tax and hence cash flow cannot be computed as tax rate  $T$  is not predictable. Further as  $K$  is not predictable, present value cannot be computed as well. So, we use simulation here.

Simulation process gives a probability distribution to each of the truant playing variables. Let the probability distribution for  $T$  and  $K$  be as follows:

$T$		$K$	
Probability	Value	Probability	Value
0.20	30%	0.30	10%
0.50	35%	0.50	11%
0.30	40%	0.20	12%

Next, we construct cumulative probability and assign random number ranges as shown in Table 11.6 separately for  $T$  and  $K$ . Two digit random number ranges are used. We start with 00 and end with 99, thus using 100 random numbers. For the different values of the variable in question, as many number of random number as are equal by the probability values of respective values are used. Thus, for variable  $T$ , 20% of random numbers aggregated for its first value 30% and 50% of random number for its next value 35% and 40%.

**Table 11.6** Cumulative Probability and Random Number Range

$T$				$K$			
Value	Probability	Cumulative Probability	Random Number Range	Value	Probability	Cumulative Probability	Random Number Range
30%	0.20	0.20	00-19	10%	0.30	0.30	00-29
35%	0.50	0.70	20-69	11%	0.50	0.80	30-79
40%	0.30	1.00	70-99	12%	0.20	1.00	80-99

For the first value of the unpredictable variable, we assign random number 00 to 19. For the second value we assign random numbers 20 – 69 and for the third value, 70–99 are assigned. Similarly for the variable  $K$  random numbers are assigned.

Simulation process now involves reading from random number table, random number pairs (one for  $T$  and another for  $K$ ). The values of  $T$  and  $K$  corresponding to the random numbers read are taken from the above table. Suppose the random numbers read are: 48 and 80. Then  $T$  is 35% as the random number 48 falls in the random number range 20-69 corresponding to 35% and  $K$  is 12% as the random number 80 falls in the random number range 80-99 corresponding to 12%. Now taking the  $T = 35\%$  and  $K = 12\%$ , the NPV of the project can be worked out.

We know that the project gives a PBT of ₹ 9,00,000 p/a for 3 years. So, the PAT = 9,00,000 – Tax @ 35% = ₹ 9,00,000 – 3,15,000 = ₹ 5,85,000 p.a. To this we have to add depreciation ₹ 6,00,000 (i.e., ₹ 18,00,000 / 3 years) to get the cash flow. So, the cash flow = 5,85,000 + 6,00,000 = ₹ 11,85,000 p.a.

$$\begin{aligned} \text{NPV} &= \sum_{t=1}^n \text{CF}_t / (1+K)^t - I \\ &= (11,85,000/1.12 + 11,85,000/1.12^2 + 11,85,000/1.12^3) - 18,00,000 \\ &= 11,85,000 [1/1.12 + 1/1.12^2 + 1/1.12^3] - 18,00,000 \\ &= 11,85,000 \times 2.4018 - 18,00,000 \\ &= 28,56,798 - 18,00,000 = ₹ 10,56,798 \end{aligned}$$

We have just taken one pair of random numbers from the table and calculated that the NPV as ₹ 10,56,798.

This process must be repeated at least 20 times, reading 20 pairs of random numbers and getting the NPV for values of T and K corresponding to each pair of random numbers read. Suppose the next pair of random numbers is 28 and 49. Corresponding T = 35% and K = 11%. Then the PAT = PBT – T = 9,00,000 – 3,15,000 = 5,85,000. The cash flow = 5,85,000 + 6,00,000 = ₹ 11,85,000.

$$\begin{aligned} \text{NPV} &= \sum_{t=1}^n \text{CF}_t / (1+K)^t - I \\ &= (11,85,000/1.11 + 11,85,000/1.11^2 + 11,85,000/1.11^3) - 18,00,000 \\ &= (10,67,598 + 9,61,773 + 8,66,462) - 18,00,000 \\ &= 28,95,803 - 18,00,000 = ₹ 10,95,803 \end{aligned}$$

Similarly the NPV for other simulations can be obtained. Thus computed NPVs may be averaged and if the same is positive, the project may be selected.

### Check Your Progress

1. Define random number.
2. Define pseudorandom number.
3. Define simulation. Why is it used?
4. What is the Monte Carlo technique?
5. Why is a table of random numbers used?

## 11.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Random number is a number whose probability of occurrence is the same as that of any other number in the collection.

### NOTES

## NOTES

2. Random numbers are called pseudorandom numbers when they are generated by some deterministic process and they qualify the predetermined statistical test for randomness.
3. The representation of reality in some physical form or in some form of mathematical equations may be called as simulation, i.e., simulation is imitation of reality. This is used because one is satisfied with suboptimal results or decision-making and also representation by a mathematical model is beyond the capabilities of the analyst.
4. It is a simulation technique in which statistical distribution functions are created by using a series of random numbers. This is generally used to solve problems which cannot be adequately represented by the mathematical models.
5. A table of random numbers is used for developing the simulation worksheet.

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### 11.6 SUMMARY

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- Simulation is the representation of reality through the use of a model or device which will react in the same manner as reality under a given set of conditions.
- There are two types of simulation: analog and digital or computer.
- Monte Carlo simulation is a modern management tool which is used for solving probabilistic problems.
- Three distinct models of Monte Carlo simulation are applied in three areas of application, namely queuing theory, inventory control and production line.

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### 11.7 KEY WORDS

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- **Random number:** Refers to a number assigned to a random variable following uniform probability density function
- **Pseudorandom numbers:** Random numbers that are generated by some deterministic process but have already qualified the predetermined statistical test for randomness

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### 11.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

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#### Short Answer Questions

1. When is simulation used? What is done in a simulation?



2. Differentiate between a random number and a pseudorandom number.
3. When can we resort to the Monte Carlo technique?
4. Who gave the code name Monte Carlo?

### Long Answer Questions

1. Explain simulation and its types.
2. What is a random variable? Why is it used?
3. Explain the methodology of the Monte Carlo simulation techniques.
4. The following data is observed in a tea serving counter. The arrival is for one minute interval.

No. of Persons Arriving	0	1	2	3	4	5
Probability	0.05	0.15	0.40	0.20	0.15	0.05

The service is taken as 2 person for one-minute interval. Using the following random numbers simulate for 15-minutes period.

09, 54, 94, 01, 80, 73, 20, 26, 90, 79, 25, 48, 99, 25, 89

Calculate also the average number of persons waiting in queue per minute.

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## 11.9 FURTHER READING

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- Taha, H.A. 2006. *Operations Research: An Introduction*, 8th edition. New Delhi: Prentice-Hall of India.
- Jensen, Paul A., and Jonathan F. Bard. 2003. *Operations Research Models and Methods*. New York: John Wiley & Sons.
- Gupta, P.K., and D.S. Hira. 2002. *Introduction to Operations Research*. New Delhi: S. Chand & Co.
- Kothari, C.R. 1982. *An Introduction to Operational Research*. New Delhi: Vikas Publishing House Pvt. Ltd.
- Kalavathy, S. 2013. *Operations Research*, 4th edition. New Delhi: Vikas Publishing House Pvt. Ltd.

### NOTES

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**BLOCK - IV**

**QUEUEING AND DECISION TREE ANALYSIS**

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**UNIT 12 QUEUEING THEORY**

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**Structure**

- 12.0 Introduction
- 12.1 Objectives
- 12.2 Queueing System: Introduction and Definitions
- 12.3 Queueing Theory: M/M/1 Queueing Model and Applications
- 12.4 New Product Launch Problems using Monte Carlo Simulation
- 12.5 Answers to Check Your Progress Questions
- 12.6 Summary
- 12.7 Key Words
- 12.8 Self Assessment Questions and Exercises
- 12.9 Further Reading

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**12.0 INTRODUCTION**

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In this unit, you will understand the basics of queueing theory. We find people standing in queue to take some services from some service facilities. Such queues are found at the counters for booking railway tickets, at bus stands, etc. After entering a service centre, if a person is serviced before arrival of another person, no queue is formed. But if persons arrive while service provider is busy serving a customer, they have to wait and a queue or waiting line is formed. Queueing theory carries out study of waiting lines using mathematical models.

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**12.1 OBJECTIVES**

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After going through this unit, you will be able to:

- Describe queueing theory
- Understand queueing discipline
- Know about customer behaviour
- Define queueing system
- Define various mathematical models
- Solve problems on queueing theory and M/M/1 model

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## 12.2 QUEUEING SYSTEM: INTRODUCTION AND DEFINITIONS

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A flow of customers from finite/infinite population towards the service facility forms a *queue* (waiting line). In the absence of a perfect balance between the service facilities and the customers, waiting time is required either for the service facilities or for the customer's arrival.

The **arrival unit** that requires some service to be performed is called *customer*. The customer may be persons, machines, vehicles, etc. Queue (waiting line) stands for the number of customers waiting to be serviced. This does not include the customer being serviced. The process or system that provides services to the customer is termed as service channel or service facility.

A queueing system can be completely described by,

- (i) The input (arrival pattern)
- (ii) The service mechanism (service pattern)
- (iii) The queue discipline
- (iv) Customer's behaviour

### Input (arrival pattern)

Input describes the way in which the customers arrive and join the system. Generally, customers arrive in a more or less random fashion, which is not worth predicting. Thus, the arrival pattern can be described in terms of probabilities and consequently the probability distribution for inter-arrival times (the time between two successive arrivals) must be defined. We deal with those queueing systems in which the customers arrive in Poisson fashion. The mean arrival rate is denoted by  $\lambda$ .

### Service mechanism

This means, the arrangement of service facility to serve customers. If there is an infinite number of servers, then all the customers are served instantaneously on arrival and there will be no queue.

If the number of servers is finite then the customers are served according to a specific order with service time as a constant or random variable. Distribution of service time that is important in practice is the *negative exponential distribution*. The **mean service rate** is denoted by  $\mu$ .

### Queue discipline

It is a rule according to which the customers are selected for service when a queue is formed. The most common disciplines are:

- First Come First Served (FCFS)
- First In First Out (FIFO)

## NOTES

- Last In First Out (LIFO)
- Selection for Service In Random Order (SIRO)

## NOTES

There are various other disciplines according to which a customer is served in preference over the other. Under priority discipline, the service is of two types, namely pre-emptive and non pre-emptive. In pre-emptive system, the high priority customers are given service over the low priority customers; in non pre-emptive system, a customer of low priority is serviced before a customer of high priority. In the case of parallel channels 'fastest server rule' is adopted.

### Customer's Behaviour

The customers generally behave in the following four ways:

- Balking:** A customer who leaves the queue because the queue is too long and he has no time to wait or does not have sufficient waiting space.
- Reneging:** This occurs when a waiting customer is impatient and leaves the queue.
- Priorities:** In certain applications some customers are served before others, regardless of their arrival. These customers have priority over others.
- Jockeying:** Customers may jockey from one waiting line to another. This is most common in a supermarket.

### Transient and steady states

A system is said to be in a *transient state* when its operating characteristics are dependent on time.

A steady state system is the one in which the behaviour of the system is independent of time. Let  $P_n(t)$  denote the probability that there are  $n$  customers in the system, at time  $t$ . Then in steady state,

$$\begin{aligned} & \lim_{t \rightarrow \infty} P_n(t) = P_n \quad (\text{Independent of } t) \\ \Rightarrow & \frac{dP_n(t)}{dt} = \frac{dP_n}{dt} \\ \Rightarrow & \lim_{t \rightarrow \infty} P'_n(t) = 0 \end{aligned}$$

**Traffic intensity or utilization factor:** An important measure of a simple queue is its traffic intensity and is given by,

$$\text{Traffic Intensity, } \rho = \frac{\text{Mean arrival rate}}{\text{Mean service rate}} = \frac{\lambda}{\mu}$$

The unit of traffic intensity is *Erlang*.

## 12.3 QUEUEING THEORY: M/M/1 QUEUEING MODEL AND APPLICATIONS

The queueing models are classified as follows:

**Model I [(M/M/1): (∞/FCFS)]:** This denotes Poisson arrival (exponential inter-arrival), Poisson departure (exponential service time), Single server, Infinite capacity and First Come First Served or service discipline. The letter *M* is used due to Markovian property of exponential process.

**Model II [(M/M/1): (N/FCFS)]:** In this model, the capacity of the system is limited (finite), say *N*. Obviously, the number of arrivals will not exceed the number *N* in any case.

**Model III [(M/M/S):(∞/FCFS)]:** This model takes the number of service channels as *S*.

**Model IV: [(M/M/S): (N/FCFS)]:** This model is essentially the same as model II, except the maximum number of customers in the system is limited to *N*, where, (*N* > *S*).

### Model I: (M/M/1) (∞/FCFS)–Birth and Death Model

**To obtain the steady state equations:** The probability that there will be *n* units (*n* > 0) in the system at time (*t* + Δ*t*), may be expressed as the sum of three independent compound probabilities by using the fundamental properties of probability, Poisson arrivals and exponential service times.

The following are the three cases:

Time ( <i>t</i> ) No. of Units	Arrival	Service	Time ( <i>t</i> + Δ <i>t</i> ) No. of Units
<i>n</i> - 1	0	0	<i>n</i>
<i>n</i> - 1	1	0	<i>n</i>
<i>n</i> + 1	0	1	<i>n</i>

Now, by adding the above three independent compound probabilities, we obtain the probability of *n* units in the system at time (*t* + Δ*t*).

$$P_n(t + \Delta t) = P_n(t) (1 - (\lambda + \mu) \Delta t) + P_{n-1}(t) \lambda \Delta t + P_{n+1}(t) \mu \Delta t + 0(\Delta t)$$

$$\frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = -(\lambda + \mu) P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) + \frac{0(\Delta t)}{\Delta t}$$

$$\lim_{\Delta t \rightarrow 0} \frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \left[ -(\lambda + \mu) P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) + \frac{0(\Delta t)}{\Delta t} \right]$$

$$\frac{dP_n(t)}{dt} = -(\lambda + \mu) P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t)$$

$$\left( \because \lim_{\Delta t \rightarrow 0} \frac{0(\Delta t)}{\Delta t} = 0 \right)$$

Where, *n* > 0

### NOTES

In the steady state,

$$P_n(t) \rightarrow 0, P_n(t) = P_n$$

$$0 = -(\lambda + \mu)P_n + \lambda P_{n-1} + \mu P_{n+1} \quad (12.1)$$

## NOTES

In a similar fashion, the probability that there will be  $n$  units (i.e.,  $n=0$ ) in the system at time  $(t+\Delta t)$  will be the sum of the following two independent probabilities.

(i) Probability that there is no unit in the system at time  $t$  and no arrival in time  $\Delta t$

$$= P_0(t) (1 - \lambda \Delta t)$$

(ii) Probability that there is one unit in the system at time  $t$ , one unit serviced in  $\Delta t$  and no arrival in  $\Delta t$

$$= P_1(t) \mu \Delta t (1 - \lambda \Delta t)$$

$$= P_1(t) \mu \Delta t - 0(\Delta t)$$

Adding these two probabilities,

$$P_0(t + \Delta t) = P_0(t)(1 - \lambda \Delta t) + P_1(t) \mu \Delta t + 0(\Delta t)$$

$$\frac{P_0(t + \Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) + \mu P_1(t) + \frac{0(\Delta t)}{\Delta t}$$

$$\lim_{\Delta t \rightarrow 0} \frac{P_0(t + \Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) + \mu P_1(t), \quad \text{for } n = 0$$

$$\frac{dP_0(t)}{d(t)} = -\lambda P_0(t) + \mu P_1(t)$$

Under steady state, we have,

$$0 = -\lambda P_0 + \mu P_1 \quad (12.2)$$

Equations (12.1) and (12.2) are called *steady state difference equations* for this model.

From Equation (12.2),  $P_1 = \frac{\lambda}{\mu} P_0$

From Equation (12.1),  $P_2 = \frac{\lambda}{\mu} P_1 = \left(\frac{\lambda}{\mu}\right)^2 P_0$

Generally,  $P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$

Since,  $\sum_{n=0}^{\infty} P_n = 1$

$$\Rightarrow P_0 + \frac{\lambda}{\mu} P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0 + \dots = 1$$

$$P_0 \left[ 1 + \frac{\lambda}{\mu} + \left( \frac{\lambda}{\mu} \right)^2 + \dots \right] = 1$$

i.e., 
$$P_0 \left( \frac{1}{1 - \frac{\lambda}{\mu}} \right) = 1$$

Since,  $\frac{\lambda}{\mu} < 1$ , hence sum of infinite geometric progression is valid.

$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \rho \quad \left( \because \rho = \frac{\lambda}{\mu} \right)$$

Also, 
$$P_n = \left( \frac{\lambda}{\mu} \right)^n P_0 = \left( \frac{\lambda}{\mu} \right)^n \left( 1 - \frac{\lambda}{\mu} \right)$$

$$P_n = \rho^n (1 - \rho)$$

### Measures of Model I

1. Expected (average) number of units in the system  $L_S$ ,

$$\begin{aligned} L_S &= \sum_{n=1}^{\infty} n P_n \\ &= \sum_{n=1}^{\infty} n \left( \frac{\lambda}{\mu} \right)^n \left( 1 - \frac{\lambda}{\mu} \right) \\ &= \left( 1 - \frac{\lambda}{\mu} \right) \frac{\lambda}{\mu} \sum_{n=1}^{\infty} n \left( \frac{\lambda}{\mu} \right)^{n-1} \\ &= \left( 1 - \frac{\lambda}{\mu} \right) \frac{\lambda}{\mu} \left( 1 + 2 \left( \frac{\lambda}{\mu} \right) + 3 \left( \frac{\lambda}{\mu} \right)^2 + \dots \right) \\ &= \left( 1 - \frac{\lambda}{\mu} \right) \left( \frac{\lambda}{\mu} \right) \left( 1 - \frac{\lambda}{\mu} \right)^{-2} \\ &= \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}} = \frac{\rho}{1 - \rho}, \rho = \frac{\lambda}{\mu} < 1 \end{aligned}$$

$$L_S = \frac{\rho}{1 - \rho}$$

2. Expected (average) queue length  $L_q$ ,

$$L_q = L_S - \frac{\lambda}{\mu} = \frac{\rho^2}{1 - \rho}$$

$$L_q = \frac{\rho^2}{1 - \rho}$$

### NOTES

## NOTES

3. Expected waiting line in the queue
- $W_q$
- ,

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu-\lambda}$$

4. Expected waiting line in the system
- $W_S$
- ,

$$\begin{aligned} W_S &= W_q + \frac{1}{\mu} \\ &= \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu} = \frac{1}{\mu-\lambda} \end{aligned}$$

$$W_S = \frac{1}{\mu-\lambda}$$

5. Expected waiting time of a customer who has to wait (
- $W/W > 0$
- ),

$$(W/W > 0) = \frac{1}{\mu-\lambda} = \frac{1}{\mu(1-\rho)}$$

6. Expected length of non-empty queue (
- $L/L > 0$
- ),

$$(L/L > 0) = \frac{\mu}{\mu-\lambda} = \frac{1}{1-\rho}$$

7. Probability of queue size
- $\geq N = \rho^N$

8. Probability of waiting time in the queue or system
- $\geq t = \int_t^{\infty} \rho(\mu-\lambda)e^{-(\mu-\lambda)\omega} d\omega$
- ,

where,  $\omega$  denotes time.

9. Traffic intensity,
- $\rho = \frac{\lambda}{\mu}$

**Inter-relationship between  $L_S, L_q, W_S, W_q$** 

We know,

$$L_S = \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}} = \frac{\lambda}{\mu - \lambda}$$

$$W_S = \frac{1}{\mu - \lambda}$$

$\therefore$

$$L_S = \lambda W_S$$



Similarly,  $L_q = \lambda W_q$  hold in general,

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$W_S = \frac{1}{\mu - \lambda}$$

$$\therefore W_S - \frac{1}{\mu} = \frac{1}{\mu - \lambda} - \frac{1}{\mu} = \frac{\mu - (\mu - \lambda)}{\mu(\mu - \lambda)} = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$\therefore W_q = W_S - \frac{1}{\mu}$$

Multiplying both sides by  $\lambda$ , we have,

$$\lambda W_q = \lambda \left( W_S - \frac{1}{\mu} \right)$$

$$L_q = \lambda W_S - \frac{\lambda}{\mu} = L_S - \frac{\lambda}{\mu}$$

$$L_q = L_S - \frac{\lambda}{\mu}$$

**Example 12.1:** A television mechanic finds that the time spent on his jobs has an exponential distribution with mean 30 minutes, if he repairs sets in the order in which they come in. If the arrival of sets is approximately Poisson with an average rate of 10 per eight-hour day then, what is the mechanic's expected idle time each day? How many jobs are ahead of the average set just brought in?

**Solution:** Here,

Mean service rate per minute,  $\mu = 1/30$ ,

Mean arrival rate per minute,

$$\lambda = \frac{10}{8 \times 60} = \frac{1}{48}$$

Expected number of jobs are,

$$\begin{aligned} L_S &= \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}} = \frac{\lambda}{\mu - \lambda} \\ &= \frac{\frac{1}{48}}{1/30 - 1/48} = 11\frac{2}{3} \text{ jobs.} \end{aligned}$$

Since, the fraction of the time, the mechanic is busy equals to  $\frac{\lambda}{\mu}$ , the number of hours for which the repairman remains busy in an eight-hour day,

$$= 8 \left( \frac{\lambda}{\mu} \right) = 8 \times \frac{30}{48} = 5 \text{ hours}$$

## NOTES

Therefore, the time for which the machanic remains idle in an eight-hour day =  $(8 - 5)$  hours = 3 hours.

Expected number of T.V. sets for repair in the system, on the average is given

**NOTES**

by  $L_S$ . It is  $\frac{5}{3} \approx 2$ .

**Example 12.2:** At what average rate must a clerk at a supermarket work in order to ensure a probability of 0.90 that the customers will not have to wait longer than 12 minutes? It is assumed that there is only one counter to which customers arrive in a Poisson fashion at an average rate of 15 per hour. The length of service by the clerk has an exponential distribution.

**Solution:** Here,

Mean arrival rate,  $\lambda = \frac{15}{60} = \frac{1}{4}$  customer/minute,

Mean service rate,  $\mu = ?$

Probability (waiting time  $\geq 12$ ) =  $1 - 0.9 = 0.10$

$$\text{Hence, } P(t \geq 12) = \int_{12}^{\infty} \rho \left(1 - \frac{\mu}{\lambda}\right) e^{-(\mu-\lambda)\omega} d\omega = 0.10$$

$$= \frac{\lambda}{\mu} (\mu - \lambda) \left( \frac{e^{-(\mu-\lambda)\omega}}{-(\mu-\lambda)} \right)_{12}^{\infty} = 0.10$$

$$\frac{\lambda}{\mu} (e^{-12(\mu-\lambda)}) = 0.10$$

$$e^{(3-12\mu)} = 0.4\mu$$

$$\mu = 0.4023$$

$$\frac{1}{\mu} = 2.49 \text{ minutes per service.}$$

**Example 12.3:** Arrivals at a telephone booth are considered to be Poisson with an average time of 10 minutes between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean three minutes.

- (i) What is the probability that a person arriving at the booth will have to wait?
- (ii) What is the average length of the queue that forms from time to time?
- (iii) The telephone department will install a second booth when convinced that an arrival would have to wait at least three minutes for the phone. By how much must the flow of arrivals be increased in order to justify a second booth?

**Solution:** Given,  $\lambda = 1/10$ ,  $\mu = 1/3$

$$(i) \quad \text{Probability } (W > 0) = 1 - P_0 = \frac{\lambda}{\mu} = \frac{1}{10} \times \frac{3}{1} = 3/10 = 0.3$$

(ii)  $(L/L > 0) = \frac{\mu}{\mu - \lambda} = \frac{1/3}{1/3 - 1/10} = 1.43 \text{ Persons} \simeq 2 \text{ Persons}$

(iii)  $W_q = \frac{\lambda}{\mu(\mu - \lambda)}$

Let arrival rate for second booth be  $\lambda'$

Since,  $W_q = 3, \mu = \frac{1}{3}, \lambda = \lambda'$  for second booth,

$$3 = \frac{\lambda'}{\frac{1}{3}\left(\frac{1}{3} - \lambda'\right)} \Rightarrow \lambda' = 0.16$$

Hence, increase in the arrival rate =  $0.16 - 0.10 = 0.06$  arrival per minute.

**Example 12.4:** As in example 12.3, in a telephone booth with Poisson arrivals spaced 10 minutes apart on the average and exponential call length averaging three minutes.

- (i) What is the probability that an arrival will have to wait for more than 10 minutes before the phone becomes free?
- (ii) What is the probability that it will take him more than 10 minutes in total to wait for the phone and complete his call?
- (iii) Estimate the fraction of a day that the phone will be in use.
- (iv) Find the average number of units in the system.

**Solution:** Given,  $\lambda = 0.1$  arrival/minute  
 $\mu = 0.33$  service/minute

$$\rho = \frac{\lambda}{\mu} = \frac{1/10}{1/3} = 0.3$$

(i) Probability (waiting time  $\geq 10$ )  $= \int_{13}^{\infty} \left(1 - \frac{\lambda}{\mu}\right) \lambda e^{-(\mu-\lambda)\omega} d\omega$

Waiting time for more than 10 minutes in queue means waiting time for more than  $10 + \frac{1}{\mu} = 10 + 3 = 13$  minutes in the systems.

Hence,

$$= - \frac{\lambda}{\mu} \left( e^{-(\mu-\lambda)\omega} \right)_{13}^{\infty}$$

$$= 0.3 e^{-2.99} = 0.015$$

- (ii) Probability (waiting time in the system  $\geq 10$ )

**NOTES**

**NOTES**

$$= \int_{10}^{\infty} \rho(\mu - \lambda)e^{-(\mu - \lambda)\omega} d\omega$$

$$= \rho e^{-10(\mu - \lambda)} = 0.3e^{-2.3} = 0.03$$

(iii) The fraction of a day that the phone will be busy = Traffic intensity

$$\rho = \frac{\lambda}{\mu} = 0.3$$

(iv) Average number of units in the system,

$$L_S = \frac{\lambda}{\mu - \lambda} = \frac{1}{\frac{1}{3} - \frac{1}{10}} = 3/7 = 0.43 \approx 1 \text{ customer.}$$

**Model II [(M/M/1): (N/FCFS)]**

This model differs from Model I in the sense that, the maximum number of customers in the system is limited to  $N$ . Therefore, the difference equations of Model I are valid for this model as long as  $n < N$ . Arrivals will not exceed  $N$  in any case. The various measures of this model are,

1.  $P_0 = \frac{1 - \rho}{1 - \rho^{N+1}}$  where,  $\rho = \frac{\lambda}{\mu}$  ( $\frac{\lambda}{\mu} > 1$  is allowed)
2.  $P_N = \frac{1 - \rho}{1 - \rho^{N+1}} \rho^N$ , for  $n = 0, 1, 2, \dots, N$
3.  $L_S = P_0 \sum_{n=0}^N n \rho^n$
4.  $L_q = L_S - \frac{\lambda}{\mu}$
5.  $W_S = L_S / \lambda$
6.  $W_q = L_q / \lambda$

**Example 12.5:** In a railway marshalling yard, goods trains arrive at the rate of 30 trains per day. Assume that the inter-arrival time follows an exponential distribution and the service time is also to be assumed as exponential with mean of 36 minutes. Calculate,

- (i) The probability that the yard is empty.
- (ii) The average queue length assuming that the line capacity of the yard is nine trains.

**Solution:** For Model II

Given,  $\lambda = \frac{30}{60 \times 24} = \frac{1}{48}$ ,  $\mu = \frac{1}{36}$

$$\rho = \frac{\lambda}{\mu} = \frac{36}{48} = 0.75$$

- (i) The probability that the queue is empty is given by,

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}, \text{ where } N=9$$

$$= \frac{1-0.75}{1-(0.75)^{9+1}} = \frac{0.25}{0.94} = 0.266$$

(ii) Average queue length is given by,

$$L_s = \frac{1-\rho}{1-\rho^{N+1}} \sum_{n=0}^N n\rho^n$$

$$= \frac{1-0.75}{1-(0.75)^{10}} \sum_{n=0}^9 n(0.75)^n$$

$$= 0.266 \times 9.58 = 2.55 \text{ trains} \simeq 3 \text{ trains}$$

(Approx.)

**Example 12.6:** A barber shop has space to accommodate only 10 customers. He can serve only one person at a time. If a customer comes to his shop and finds it full, he goes to the next shop. Customers randomly arrive at an average rate  $\lambda = 10$  customers per hour and the barber's service time is negative exponential with an average of  $1/\mu = 5$  minutes per customer. Find  $P_0$  and  $P_n$ .

**Solution:** Given,  $N = 10$ ,  $\lambda = \frac{10}{60}$ ,  $\mu = \frac{1}{5}$

$$\rho = \frac{\lambda}{\mu} = \frac{5}{6}$$

$$P_0 = \frac{1-\rho}{1-\rho^{11}} = \frac{1-5/6}{1-(5/6)^{11}}$$

$$= \frac{0.1667}{0.8655} = 0.1926$$

$$P_n = \left( \frac{1-\rho}{1-\rho^{N+1}} \right) \rho^n$$

$$= (0.1926) \times \left( \frac{5}{6} \right)^N = 0.1926 \left( \frac{5}{6} \right)^{10} \quad [\because N=10]$$

**Example 12.7:** A car park contains 5 cars. The arrival of cars is Poisson at a mean rate of 10 per hour. The length of time each car spends in the car park is exponential distribution with mean of 0.5 hours. How many cars are in the car park on an average?

**Solution:** Given,  $N = 5$ ,  $\lambda = \frac{10}{60} = \frac{1}{6}$ ,  $\mu = \frac{0.5}{60} = \frac{1}{2 \times 60} = \frac{1}{120}$

$$\rho = \frac{\lambda}{\mu} = 20$$

## NOTES

$$P_0 = \left( \frac{1-\rho}{1-\rho^{N+1}} \right) = \frac{1-20}{1-20^6} = \frac{-19}{-63999999} = 2.9692 \times 10^{-7}$$

$$L_s = P_0 \sum_{n=0}^N n\rho^n = (2.9692 \times 10^{-7}) \sum_{n=0}^5 n(20)^n = 6.6587 \approx 7 \text{ (Approx.)}$$

**NOTES**

**12.4 NEW PRODUCT LAUNCH PROBLEMS USING MONTE CARLO SIMULATION**

Headen Ltd. plans to introduce a new device for automobiles to warn the driver of the proximity of the car in front. Two different engineering strategies to develop the product are there. These two have different probability distributions of development time as follows:

<i>Development Time in Months</i>	<i>Strategy I</i>	<i>Strategy II</i>
6	0.21	0.42
9	0.38	0.35
12	0.41	0.23

Strategy I will require ₹ 6,00,000 in investment and will result in a variable cost of ₹ 7.5 per unit. For strategy II the respective figures are ₹ 15,00,000 and ₹ 6.75. The product will sell for ₹ 10. The sales volume depends on development time, with the following probability distribution:

Unit Sales Volume	Development Time		
	6	9	12
10,00,000	0.21	0.38	0.51
15,00,000	0.79	0.62	0.49

Simulate 5 trials for introducing the new product for each engineering strategy taking the following random numbers: 49, 67, 06, 30 and 95 for Strategy I and 01, 10, 70, 80 and 66 for Strategy II for both development time and sales volume and find the worth of the strategies.

First we have to construct cumulative probability distribution and random number range for development time strategywise, for the two strategies.

Development Time	Strategy I			Strategy II		
	P	Cu.P	R. Nos	P	Cu.P	R. Nos
6	0.21	0.21	00-20	0.42	0.42	00-41
9	0.38	0.59	21-58	0.35	0.77	42-86
12	0.41	1.00	59-99	0.23	1.00	77-99

Similarly, cumulative probability and random number range for sales volume is needed. The same is given below.

Sales Volume	Development Time								
	6 months			9 months			12 months		
	P	Cu.P	R.Nos	P	Cu.P	R.Nos	P	Cu.P	R.Nos
10,00,000	0.21	0.21	00-20	0.38	0.38	00-37	0.51	0.51	00-50
15,00,000	0.79	1.00	21-99	0.62	1.00	38-99	0.49	1.00	51-99

## NOTES

Now simulation runs need to be done. We have to read R.Nos. from the table if these are not given. But we are given the random numbers, strategywise. So, we take them.

Strategy I : R.Nos. 49, 67, 06, 30 and 95

Development time and corresponding sales volume for given R.Nos. are as follows:

R. Nos.	Development time	Sales Volume
49	9	15,00,000
67	12	15,00,000
06	6	10,00,000
30	9	10,00,000
95	12	15,00,000

Similarly for strategy II we can get the simulated development time and corresponding sales volume given the R.Nos.

Strategy II: R.Nos. 01, 10, 70, 80 and 66.

R. Nos.	Development Time	Sales Volume
01	6	10,00,000
10	6	10,00,000
70	9	15,00,000
80	12	15,00,000
66	9	15,00,000

Now the worth of the two strategies can be evaluated. The following index is used.

$$\text{Worth} = [(\text{Selling price} - \text{Variable cost}) \times \text{Volume}] / \text{Investment}$$

The average of worth index figures can be taken as measures of performance.

### Strategy I

$$\text{Selling price} - \text{Variable cost} = ₹ 10 - 7.5 = ₹ 2.5.$$

$$\text{Investment} = ₹ 6,00,000$$

Volume varies from run to run. So, worth also varies run to run.

$$\text{Worth Run 1} = (2.5 \times 15,00,000) / 6,00,000 = 6.25$$

$$\text{Run 2} = (2.5 \times 15,00,000) / 6,00,000 = 6.25$$

$$\text{Run 3} = (2.5 \times 10,00,000) / 6,00,000 = 4.16$$

$$\text{Run 4} = (2.5 \times 10,00,000) / 6,00,000 = 4.17$$

$$\begin{aligned} \text{Run 5} &= (2.5 \times 15,00,000) / 6,00,000 = 6.25 \\ \text{Total} &= 27.08 \\ \text{Average} &= 5.416 \end{aligned}$$

**NOTES**

**Strategy II**

Selling price – Variable cost = ₹ 10 – 6.75 = ₹ 3.25.

Investment = ₹ 15,00,000

Volume varies from run to run.

Worth Run 1 =  $(3.75 \times 10,00,000) / 15,00,000 = 2.5$

Run 2 =  $(3.75 \times 10,00,000) / 15,00,000 = 2.5$

Run 3 =  $(3.75 \times 15,00,000) / 15,00,000 = 3.75$

Run 4 =  $(3.75 \times 15,00,000) / 15,00,000 = 3.75$

Run 5 =  $(3.75 \times 15,00,000) / 15,00,000 = 3.75$

Total = 16.25

Average = 3.25

**Strategy I is worthier than Strategy II**

**Example 12.8:** A trader has the following probability receipts and monthly cash expenses.

Receipts (₹ lakhs)	:	15	16	17	18	19
		20				
Probability	:	0.30	0.25	0.15	0.15	0.10
		0.05				
Expenses (₹ lakhs)	:	13	14	15	16	17
		18				
Probability	:	0.15	0.20	0.25	0.20	0.15
		0.05				

The trader has an opening balance of ₹ 2 lakhs. Assume overdraft facility as available which has to be cleared at the next opportunity. Find the lowest level, highest level and closing level of cash during the year.

**Solution:**

**Step 1:** Since monthly figures are given and annual position is asked, at least 12 runs must be made. For these 12 runs, let the R.Nos. pairs for receipts–payment combination be: 15:48, 20:98, 73:06, 60:45, 44:15, 18:19, 58:15, 61:67, 18:90, 00:58, 32:68 and 65:73.



**Step 2: Cumulative probability and R.No. assignment**

Receipts				Payments			
Value	P	Cu.P	R.No	Value	P	Cu.P	R.No
15	0.30	0.30	00-29	13	0.15	0.15	00-14
16	0.25	0.55	30-54	14	0.20	0.35	15-34
17	0.15	0.70	55-69	15	0.25	0.60	35-59
18	0.15	0.85	70-84	16	0.20	0.80	60-79
19	0.10	0.95	85-94	17	0.15	0.95	80-94
20	0.05	1.00	95-99	18	0.05	1.00	95-99

**NOTES**

Now the simulation runs have to be made as follows:

Month	R.No. for Receipts	Corresponding Receipts	Receipts plus previous closing	R.No. for payment	Payments	Closing Balance	Overdraft
1	15	15	15 + 2 = 17	48	15	17 - 15 = 2	0
2	20	15	15 + 2 = 17	98	18	0	1
3	73	18	18 + 0 = 18	06	13 + 1	18 - 14 = 4	0
4	60	17	17 + 4 = 21	45	15	21 - 15 = 6	0
5	44	16	16 + 6 = 22	15	14	22 - 14 = 8	0
6	18	15	15 + 8 = 23	19	14	23 - 14 = 9	0
7	58	17	17 + 9 = 26	15	14	26 - 14 = 12	0
8	61	17	17 + 12 = 29	67	16	29 - 16 = 13	0
9	18	15	15 + 13 = 28	90	17	28 - 17 = 11	0
10	00	15	15 + 11 = 26	58	15	26 - 15 = 11	0
11	32	16	16 + 11 = 27	68	16	27 - 16 = 11	0
12	65	17	17 + 11 = 28	73	16	28 - 16 = 12	0

At year end, a cash balance of ₹ 12 lakhs remains. The peak point is ₹ 13 lakhs on hand during the 8th month. The lowest point is in the 2th month with overdraft of ₹ 1 lakhs. The overdraft is returned in the next month itself.

**Check Your Progress**

1. Define a queue.
2. What is FCFS in a queueing discipline?
3. What is SIRO?
4. What is jockeying?

**12.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS**

1. Number of customers waiting for service form a queue. This does not include persons or customer being served.
2. In FCFS queueing discipline, a person is given service according to their arrival. Those who arrive first are served first.

## NOTES

3. SIRO is a queueing discipline in which service is provided in random order, irrespective of arrival of customers.
4. Jockeying is a type of behaviour shown by customer when he moves from one queue to another, expecting quicker service.

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### 12.6 SUMMARY

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- Queueing discipline tells how customers are provided service while waiting in a queue. When a queue is formed and there is limited service channel, customers are served according to some queueing discipline that may be; FCFS or FIFO, LIFO or LCFS, SIRO and priority queue.
- In FCFS or LIFO, customers are served according to their order of arrival. Those who come first are attended first. LIFO follows a pattern different from that and person who comes last are served first. This discipline is found in employment where an employee that comes last is to go first.
- SIRO is a queueing discipline where service is provided in random order. Priority queue denotes a queueing discipline in which customers are given some priority depending on their position and status or value of their time. In this discipline order of arrival does not matter.
- Traffic intensity is the ratio of mean arrival rate to mean service rate. Unit of traffic intensity is Erlang. Queueing systems are represented as mathematical models that describe probability on rate of arrival, probability law according to which customers are provided services, numbers of channel for services and queueing discipline followed.
- A queueing system has essentially four components. These are; pattern in which customers arrive, service pattern followed such as number of service channels, queue discipline that is followed in serving customers and customer behaviour.
- Arrival pattern are probability of customers arriving within a time interval. Service pattern tells about ways and means to render services. It also tells about number of service channels kept for services to customers. Queue discipline tells about the order in which customers are served.
- One of the very important aspect of queueing system is customer behaviours. Some customers leave the queue when they find queue very long and asses waiting time to be very high termed as Balking.
- There are customers who stand for some time in the queue and find it difficult to wait further in the queue and leave the queue termed as Reneging.
- Sometimes, it is found that customers move from one waiting line to another when there are many service channels termed as Jockeying. This may be found in a supermarket where people move from one counter to another in hope of getting quicker service.

- A queueing system, in transient state, has time dependent operating characteristics. If behaviour of the system is not dependent on time, then it is said to be in steady state. If there is a probability of  $n$  customers in the system at any time  $t$ , then a time dependent system is denoted by  $P_n(t)$  and those independent of time is denoted as  $P_n$ .
- Mathematical model Model I [(M/M/1): ( $\infty$  / FCFS)] denotes arrival in exponential time interval, departure (after getting the service) in exponential service time, single server, infinite capacity and FCFS shows queueing discipline.
- In Model II [(M/M/1): (N / FCFS)] everything except capacity is same. In this model capacity is finite.

## NOTES

### 12.7 KEY WORDS

- **Arrival unit:** Anything that arrives for service is known as arrival unit. Arrival unit may be a person, a machine, a vehicle, etc., which comes for taking some kind of service.
- **Mean service rate:** Service time for attending a customer follows a negative exponential distribution. Mean service rate is given by number of customers serviced in a specified interval of time and is denoted as  $\mu$ .
- **Model I [(M/M/1): ( $\infty$ /FCFS)]:** This denotes Poisson arrival (exponential inter-arrival), Poisson departure (exponential service time), Single server, Infinite capacity and First Come First Served or service discipline.

### 12.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

#### Short Answer Questions

1. How queue is formed?
2. Write about arrival rates.
3. What is understood by mean service rate?
4. What is traffic intensity?
5. How a queueing model is denoted using symbols?

#### Long Answer Questions

1. People arrive at a theatre ticket centre in a Poisson distributed arrival rate of 25 per hour. Service time is constant at two minutes. Calculate,
  - (i) The mean number in the waiting line.
  - (ii) The mean waiting time.
  - (iii) Utilization factor.

## NOTES

2. At a one-man barber shop, customers arrive according to Poisson distribution, with a mean arrival rate of 5 per hour and the hair cutting time was exponentially distributed with an average hair cut taking 10 minutes. It is assumed that because of his excellent reputation, customers were always willing to wait. Calculate the following—
  - (i) Average number of customers in the shop and the average number waiting for a hair cut.
  - (ii) The per cent of time, an arrival can walk right in without having to wait.
  - (iii) The percentage of customers who have to wait before getting into the barber's chair.
3. Cars arrive at a petrol pump with exponential inter-arrival time having mean  $\frac{1}{2}$  minute. The attendant on an average takes  $\frac{1}{5}$  minute per car to supply petrol. The service time being exponentially distributed, determine,
  - (i) The average number of cars waiting to be served.
  - (ii) The average number of cars in the queue.
  - (iii) The proportion of time, for which the pump attendant is idle.
4. The mean arrival rate to a service centre is 3 per hour. The mean service time is found to be 10 minutes per service. Assuming Poisson arrival and exponential service time, find,
  - (i) The utilization factor for this service facility.
  - (ii) The probability of two units in the system.
  - (iii) The expected number of units in the system.
  - (iv) The expected time in minutes that a customer has to spend in the system.

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## 12.9 FURTHER READING

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## UNIT 13 DECISION ANALYSIS

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### Structure

- 13.0 Introduction
- 13.1 Objectives
- 13.2 Decision Analysis: Concept and Definition
  - 13.2.1 Decision-Making Process
- 13.3 Preparation of Pay-off and Loss Table
- 13.4 Answers to Check Your Progress Questions
- 13.5 Summary
- 13.6 Key Words
- 13.7 Self Assessment Questions and Exercises
- 13.8 Further Reading

### NOTES

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### 13.0 INTRODUCTION

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A decision taken by a manager has far-reaching effect on a business. Decisions are of two types: tactical and strategic. Tactical decisions affect the business in the short run whereas strategic decisions have a far-reaching effect. To take a right decision, managers resort to statistical methods to analyse factors that affect the business as a whole. You will know about deterministic and probabilistic decision models. A lot of foresight is required to take decisions with probabilistic models and for that some rule or criteria have to be followed. A decision-making process adopts EMV (Expected Monetary Value) and EOL (Expected Opportunity Loss) criteria for uncertain situations which involve risk.

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### 13.1 OBJECTIVES

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After going through this unit, you will be able to:

- Describe the concept and definitions of decision analysis
- Discuss payoff and loss tables
- Explain the decision making process
- Analyse the expected value of payoff

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### 13.2 DECISION ANALYSIS: CONCEPT AND DEFINITION

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Decision-making is an everyday process in life. It is the major role of a manager too. The decision taken by a manager has far reaching effect on the business. Right decisions will have salutary effect and the wrong one may prove to be disastrous.

## NOTES

**Decisions** may be classified into two categories, tactical and strategic. Tactical decisions are those which affect the business in the short run. Strategic decisions are those which have far reaching effect on the course of business.

These days, in every organization whether large or small, the person at the top management has to take some decision, knowing that certain events beyond his control may occur to make him regret the decision. He is uncertain as to whether or not these unfortunate events will happen. In such situations, the best possible decision can be made by the use of statistical methods. These methods try to minimize the degree to which the person is likely to regret the decision he makes for a particular problem.

Decision-making constitutes one of the highest forms of human activity.

The problem under study may be represented by a model in terms of the following elements:

- (i) **The Decision-maker.** The decision-maker is charged with the responsibility of making the decision. That is, he has to select one from a set of possible courses of action.
- (ii) **The Acts.** The acts are the alternative courses of action or strategies that are available to the decision-maker. The decision involves a selection among two or more alternative courses of action. The problem is to choose the best of these alternatives to achieve an objective.
- (iii) **Event.** Events are the occurrences which affect the achievement of the objectives. They are also called states of nature or outcomes. The events constitute a mutually exclusive and exhaustive set of outcomes, which describe the possible behaviour of the environment in which the decision is made. The decision-maker has no control over which event will take place and can only attach a subjective probability of occurrence of each.
- (iv) **Pay-off Table.** A pay-off table represents the economics of a problem, i.e., revenue and costs associated with any action with a particular outcome. It is an ordered statement of profit or costs resulting under the given situation. The pay-off can be interpreted as the outcome in quantitative form if the decision-maker adopts a particular strategy under a particular state of nature.
- (v) **Opportunity Loss Table.** An **opportunity loss** is the loss incurred because of failure to take the best possible action. Opportunity losses are calculated separately for each state of nature that might occur. Given the occurrence of a specific state of nature, we can determine the best possible action. For a given state of nature, the opportunity loss of an act is the difference between the pay-off of that act and the pay-off of the best act that could have been selected.

In any decision problem, the decision-maker has to choose from the available alternative courses of action, the one that yields the best result. If the consequences of each choice are known with certainty, the decision-maker

can easily make decisions. But in most of real life problems, the decision-maker has to deal with situations where uncertainty of the outcomes prevails.

### 13.2.1 Decision Making Process

#### Decision Making under Certainty

In this case, the decision-maker knows with certainty the consequences of every alternative or decision choice. The decision-maker presumes that only one state of nature is relevant for his purpose. He identifies this state of nature, takes it for granted and presumes complete knowledge as to its occurrence.

#### Decision-Making under Uncertainty

When the decision-maker faces multiple states of nature but he has no means to arrive at probability values to the likelihood of occurrence of these states of nature, the problem is a decision problem under uncertainty. Such situations arise when a new product is introduced in the market or a new plant is set up. In business, there are many problems of this nature. Here, the choice of decision largely depends on the personality of the decision-maker.

The following methods are available to the decision-maker in situations of uncertainty:

**(i) Maximax Criterion:** The term ‘maximax’ is an abbreviation of the phrase maximum of the maximums, an adventurous and aggressive decision-maker may choose to take the action that would result in the maximum pay-off possible. Suppose for each action there are three possible pay-offs corresponding to three states of nature as given in the following decision matrix:

State of Nature	Decisions		
	$A_1$	$A_2$	$A_3$
$S_1$	220	180	100
$S_2$	160	190	180
$S_3$	140	170	200

Maximum under each decision are (220, 190, 200). The maximum of these three maximums is 220. Consequently, according to the maximax criteria the decision to be adopted is  $A_1$ .

**(ii) Minimax Criterion:** Minimax is just the opposite of maximax. Application of the minimax criteria requires a table of losses instead of gains. The losses are the costs to be incurred or the damages to be suffered for each of the alternative actions and states of nature. The minimax rule minimizes the maximum possible loss for each course of action. The term ‘minimax’ is an abbreviation of the phrase minimum of the maximum. Under each of the various actions, there is a maximum loss and the action that is associated with the minimum of the various maximum losses is the action to be taken according to the minimax criterion. Suppose the loss table is as follows:

## NOTES

## NOTES

State of Nature	Decisions		
	$A_1$	$A_2$	$A_3$
$S_1$	0	4	10
$S_2$	3	0	6
$S_3$	18	14	0

It shows that the maximum losses incurred by the various decisions are,

$$\begin{array}{ccc} A_1 & A_2 & A_3 \\ 18 & 14 & 10 \end{array}$$

Also, the minimum among three maximums is 10 which is under action  $A_3$ . Thus according to minimax criterion, the decision-maker should take action  $A_3$ .

**(iii) Maximin Criterion:** The maximin criterion of decision-making stands for choice between alternative courses of action assuming pessimistic view of nature. Taking each act in turn, we note the worst possible results in terms of pay-off and select the act which maximizes the minimum pay-off. Suppose the pay-off table is as follows:

State of Nature	Action		
	$A_1$	$A_2$	$A_3$
$S_1$	-80	-60	-20
$S_2$	-30	-10	-2
$S_3$	30	15	7
$S_4$	75	80	25

Minimum under each decision on  $A_1, A_2$  and  $A_3$  are -80, -60 and -20, respectively for  $S_1$  state of nature.

The action  $A_3$  is to be taken according to this criterion because it is the maximum among minimums.

**(iv) Laplace Criterion:** In this method, as the decision-maker has no information about the probability of occurrence of various events, the decision-maker makes a simple assumption that each probability is equally likely. The expected pay-off is worked out on the basis of these probabilities. The act having maximum expected pay-off is selected.

**Example 13.1:** Calculate the maximum expected pay-off and the optimal act from the following data:

Events	Act		
	$A_1$	$A_2$	$A_3$
$E_1$	20	12	25
$E_2$	25	15	30
$E_3$	30	20	22



**Solution:** We associate equal probability for each event, say  $1/3$ . Expected pay-offs are:

$$A_1 \rightarrow \left(20 \times \frac{1}{3}\right) + \left(25 \times \frac{1}{3}\right) + \left(30 \times \frac{1}{3}\right) = \frac{75}{3} = 25$$

$$A_2 \rightarrow \left(12 \times \frac{1}{3}\right) + \left(15 \times \frac{1}{3}\right) + \left(20 \times \frac{1}{3}\right) = \frac{47}{3} = 15.67$$

$$A_3 \rightarrow \left(25 \times \frac{1}{3}\right) + \left(30 \times \frac{1}{3}\right) + \left(22 \times \frac{1}{3}\right) = \frac{77}{3} = 25.67$$

Since  $A_3$  has the maximum expected pay-off,  $A_3$  is the optimal act.

**(v) Hurwicz Alpha Criterion:** This method is a combination of maximum criterion and maximax criterion. In this method, the decision-maker's degree of optimism is represented by  $\alpha$ , the coefficient of optimism.  $\alpha$  varies between 0 and 1. When  $\alpha = 0$ , there is total pessimism and when  $\alpha = 1$ , there is total optimism.

Here, the criteria  $D_1, D_2, D_3$ , etc., is calculated which is connected with all strategies (state of act) and  $D_i = \alpha M_i + (1 - \alpha) m_i$  where  $M_i$  is the maximum pay-off of  $i$ th strategy and  $m_i$  is the minimum pay-off of  $i$ th strategy. The strategy with highest value of  $D_1, D_2, \dots$ , is chosen. The decision-maker will specify the value of  $\alpha$  depending upon his level of optimism.

**(vi) Savage Decision Rule:** This rule is also known as regret decision rule. It is based on general insurance against risk. It ensures against the maximum possible risk. Under it, one adopts the strategy which causes minimum of the maximum possible regrets and the given pay-off matrix is converted into a regret matrix. This is done by subtracting each entry in the pay-off matrix from the largest entry in its column. The largest entry in a column will have zero regret. Thus, in each cell we enter the difference between what the decision-maker would have done if he had known which outcome would occur, and the choice is represented by the cell. Once the regret matrix is formed, the minimax criterion can be applied to it to select the best course of action.

**Example 13.2:** Calculate the maximum and minimum pay-off from the given data to specify the value of  $\alpha$  and the act to be selected. Take  $\alpha = 0.6$ .

Events	Act		
	$A_1$	$A_2$	$A_3$
$E_1$	20	12	25
$E_2$	25	15	30
$E_3$	30	20	22

**Solution:** Given is  $\alpha = 0.6$ .

For  $A_1$ , maximum pay-off = 30

Minimum pay-off = 20

$$\therefore D_1 = (0.6 \times 30) + (1 - 0.6)20 = 26$$

$$\text{Similarly, } D_2 = (0.6 \times 20) + (1 - 0.6)12 = 16.8$$

## NOTES

$$D_3 = (0.6 \times 30) + (1 - 0.6)22 = 26.8$$

Since  $D_3$  is maximum, select the act  $A_3$ .

## NOTES

### Decision-Making under Risk

In this situation, the decision-maker has to face several states of nature. But, he has some knowledge or experience which will enable him to assign probability to the occurrence of each state of nature. The objective is to optimize the expected profit or to minimize the opportunity loss.

For decision problems under risk, the most popular methods used are EMV (Expected Monetary Value) criterion, EOL (Expected Opportunity Loss), criterion or EVPI (Expected Value of Perfect Information) criterion.

**(i) Expected Monetary Value:** When probabilities can be assigned to the various states of nature, it is possible to calculate the statistical expectation of gain for each course of action.

The conditional value of each event in the pay-off table is multiplied by its probability and the product is summed up. The resulting number is the EMV for the act. The decision-maker then selects from the available alternative actions, the action that leads to the maximum expected gain (that is the action with highest EMV). Consider the following example. Let the states of nature (Events) be  $S_1$  and  $S_2$ , and the alternative strategies (Act) be  $A_1$  and  $A_2$ . Then the pay-off table is as follows:

Events	Act	
	$A_1$	$A_2$
$S_1$	30	20
$S_2$	35	30

Let the probabilities for the states of nature  $S_1$  and  $S_2$  be respectively 0.6 and 0.4.

Then,

$$\text{EMV for } A_1 = (30 \times 0.6) + (35 \times 0.4) = 18 + 14 = 32$$

$$\text{EMV for } A_2 = (20 \times 0.6) + (30 \times 0.4) = 12 + 12 = 24$$

EMV for  $A_1$  is greater.

$\therefore$  The decision-maker will choose the strategy  $A_1$ .

**(ii) Expected Opportunity Loss:** The difference between the greater pay-off and the actual pay-off is known as **opportunity loss**. Under this criterion, the strategy which has minimum expected opportunity loss is chosen. The calculation of EOL is similar to that of EMV.

Consider the following example on opportunity loss table. Here  $A_1$  and  $A_2$  are the strategies and  $S_1$  and  $S_2$  are the states of nature.

	$A_1$	$A_2$
$S_1$	0	10
$S_2$	2	-5

Let the probabilities for two states be 0.6 and 0.4.

$$\text{EOL for } A_1 = (0 \times 0.6) + (2 \times 0.4) = 0.8$$

$$\text{EOL for } A_2 = (10 \times 0.6) + (-5 \times 0.4) = 6 - 2 = 4$$

EOL for  $A_1$  is the least. Therefore, the strategy  $A_1$  may be chosen.

**(iii) Expected Value of Perfect Information:** The expected value of perfect information is the average (expected) return in the long run, if we have perfect information before a decision is to be made.

In order to calculate EVPI, we choose the best alternative with the probability of their state of nature. The expected value of perfect information is the expected outcome with perfect information minus the outcome with maximum EMV.

$$\therefore \text{EVPI} = \text{Expected value with perfect information} - \text{Maximum EMV}$$

Consider the following example.

**Example 13.3:**  $A_1, A_2, A_3$  are the acts and  $S_1, S_2, S_3$  are the states of nature. Also,  $P(S_1) = 0.5, P(S_2) = 0.4$  and  $P(S_3) = 0.1$ . Calculate the expected value of perfect information.

**Solution:** The following is the pay-off table:

State of Nature	Pay-off Table		
	$A_1$	$A_2$	$A_3$
$S_1$	30	25	22
$S_2$	20	35	20
$S_3$	40	30	35

$$\text{EMV for } A_1 = (0.5 \times 30) + (0.4 \times 20) + (0.1 \times 40) = 15 + 8 + 4 = 27$$

$$\text{EMV for } A_2 = (0.5 \times 25) + (0.4 \times 35) + (0.1 \times 30) = 12.5 + 14 + 3 = 29.5$$

$$\text{EMV for } A_3 = (0.5 \times 22) + (0.4 \times 20) + (0.1 \times 35) = 11 + 8 + 3.5 = 22.5$$

The highest EMV is for the strategy  $A_2$  and it is 29.5.

Now to find EVPI, work out the expected value for maximum pay-off under all states of nature.

	Max. Profit of Each State	Probability	Expected Value (= Prob. $\times$ Profit)
$S_1$	30	0.5	15
$S_2$	35	0.4	14
$S_3$	40	0.1	4

$$\therefore \text{Expected pay-off with perfect information} = 33$$

## NOTES

$\therefore$  Thus, EVPI = Expected value with perfect information – Maximum EMV  
 $= 33 - 29.5 = 3.5$

## NOTES

### 13.3 PREPARATION OF PAY-OFF AND LOSS TABLE

Let us analyse the preparation of pay-off and loss tables.

**Example 13.4:** All ink manufacturers produce a certain type of ink at a total average cost of ₹ 3 per bottle and sell at a price of ₹ 5 per bottle. The ink is produced over the weekend and is sold during the following week. According to the past experience, the weekly demand has never been less than 78 or greater than 80 bottles.

You are required to formulate the pay-off table.

**Solution:** The different states of nature are the demand for 78 units, 79 units or 80 units termed as  $S_1, S_2, S_3$ , respectively.

The alternative courses of action are selling 78 units, 79 units or 80 units termed as  $A_1, A_2, A_3$ , respectively.

Selling price of ink = ₹ 5 per bottle

Cost price = ₹ 3 per bottle

Calculation of pay-offs (Pay-off stands for the gain):

Sale quantity  $\times$  Price – Production Quantity  $\times$  Cost

$$A_1S_1 = 78 \times 5 - 78 \times 3 = 390 - 234 = 156$$

$$A_2S_1 = 78 \times 5 - 79 \times 3 = 390 - 237 = 153$$

$$A_3S_1 = 78 \times 5 - 80 \times 3 = 390 - 240 = 150$$

$$A_1S_2 = 78 \times 5 - 78 \times 3 = 390 - 234 = 156$$

$$A_2S_2 = 79 \times 5 - 79 \times 3 = 395 - 237 = 158$$

$$A_3S_2 = 79 \times 5 - 80 \times 3 = 395 - 240 = 155$$

$$A_1S_3 = 78 \times 5 - 78 \times 3 = 390 - 234 = 156$$

$$A_2S_3 = 79 \times 5 - 79 \times 3 = 395 - 237 = 158$$

$$A_3S_3 = 80 \times 5 - 80 \times 3 = 400 - 240 = 160$$

(Explanation:  $A_1S_1$  means selling quantity is 78 and manufacturing quantity is 78.  $A_2S_1$  means sales 78, production 79, and so on.)

Pay-off Table

State of Nature (Events)	Act (Strategy)		
	$A_1$	$A_2$	$A_3$
$S_1$	156	153	150
$S_2$	156	158	155
$S_3$	156	158	160

**Note:** We shall show the state of nature in rows and act in columns.

**Preparation of Loss Table**

**Example 13.5:** A small ink manufacturer produces a certain type of ink at a total average cost of ₹ 3 per bottle and sells at a price of ₹ 5 per bottle. The ink is produced over the weekend and is sold during the following week. According to the past experience, the weekly demand has never been less than 78 or greater than 80 bottles in his place.

You are required to formulate the loss table.

**Solution:** Calculation of regret (opportunity loss)

$$A_1S_1 = 0 \text{ (since production and sales are of equal quantities, say 78)}$$

$$A_2S_1 = 1 \times 3 = 3 \text{ (since one unit of production is in excess whose cost = ₹ 3)}$$

$$A_3S_1 = 2 \times 3 = 6 \text{ (since 2 units of production are in excess whose unit cost is ₹ 3)}$$

$$A_1S_2 = 1 \times 2 = 2 \text{ (since the demand of one unit is more than produced, the profit for one unit is ₹ 2)}$$

Similarly,  $A_2S_2 = 0$  (Since, Units of production = Units of demand)

$$A_3S_2 = 1 \times 3 = 3 \quad A_1S_3 = 2 \times 2 = 4$$

$$A_2S_3 = 2 \times 1 = 2 \quad \text{and, } A_3S_3 = 0$$

**Opportunity Loss Table**

State of Nature (Events)	Action		
	$A_1$	$A_2$	$A_3$
$S_1$	0	3	6
$S_2$	2	0	3
$S_3$	4	2	0

From the pay-off table also, opportunity loss table can be prepared.

**Method:** Let every row of the pay-off table represent a state of nature and every column represent a course of action. Then, from each row select the highest pay-off and subtract all pay-offs of that row from it. They are the opportunity losses.

See the following examples.

**Example 13.6:** The following is a pay-off table. From it form a regret (opportunity loss) table.

State of Nature	Pay-off Table		
	$A_1$	$A_2$	$A_3$
$E_1$	156	153	150
$E_2$	156	158	155
$E_3$	156	158	160

**NOTES**

**Solution:**

**NOTES**

**Opportunity Loss Table**

<i>State of Nature</i>	<i>Action</i>		
	$A_1$	$A_2$	$A_3$
$E_1$	$156 - 156 = 0$	$156 - 153 = 3$	$156 - 150 = 6$
$E_2$	$158 - 156 = 2$	$158 - 158 = 0$	$158 - 155 = 3$
$E_3$	$160 - 156 = 4$	$160 - 158 = 2$	$160 - 160 = 0$

**Example 13.7:** The research department of a consumer products company has recommended the marketing department to launch a soap with three different perfumes. The marketing manager has to decide the type of perfume to launch under the following estimated pay-off for the various levels of sales:

**Estimated Level of Sales (Units)**

<i>Types of Perfume</i>	<b>20,000</b>	<b>10,000</b>	<b>2,000</b>
I	250	15	10
II	40	20	5
III	60	25	3

Examine which type can be chosen under maximax, minimax, maximin, Laplace and Hurwicz alpha criteria.

**Solution:** Rewriting the pay-off table,

<i>Levels of Sales</i>	<i>Types of Perfume</i>		
	<b>I</b>	<b>II</b>	<b>III</b>
20000	250	40	60
10000	15	20	25
2000	10	5	3

- (i) Maximax criterion  
 Maximum for Type I = 250  
 Maximum for Type II = 40  
 Maximum for Type III = 60  
 Maximum of maximum = 250  
 ∴ Select Type I perfume.

- (ii) Minimax criterion

Loss table is,

<i>Sales</i>	<i>Types</i>		
	<b>I</b>	<b>II</b>	<b>III</b>
20,000	0	210	190
10,000	10	5	0
2,000	0	5	7

Maximum losses under I, II, III are respectively 10, 210, 190 (from pay-off table)

Minimum of these is 10.

∴ Type I perfume is preferred.

(iii) Maximin criterion

Minimum pay-off under each Type I, II, III are respectively 10, 5, 3 (from pay-off table)

Maximum of these is 10.

∴ Type I perfume is preferred.

(iv) Laplace criterion

Let the probability for each levels of sales be taken as  $1/3$  each.

Expected pay-offs are:

$$\text{Type I} \rightarrow \left(250 \times \frac{1}{3}\right) + \left(15 \times \frac{1}{3}\right) + \left(10 \times \frac{1}{3}\right) = \frac{275}{3} = 91.67$$

$$\text{Type II} \rightarrow \left(40 \times \frac{1}{3}\right) + \left(20 \times \frac{1}{3}\right) + \left(5 \times \frac{1}{3}\right) = \frac{65}{3} = 21.67$$

$$\text{Type III} \rightarrow \left(60 \times \frac{1}{3}\right) + \left(25 \times \frac{1}{3}\right) + \left(3 \times \frac{1}{3}\right) = \frac{88}{3} = 29.33$$

Maximum expected pay-off is for Type I, so choose Type I perfume.

## NOTES

### Check Your Progress

1. How are decisions classified?
2. What are the different situations that a decision maker has to deal with?
3. How is the EMV calculated?
4. Define opportunity loss.
5. What is expected value with perfect information?

## 13.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. The decision can be classified as:
  - (i) Tactical decision
  - (ii) Strategic decision
2. A decision maker has to deal with the following situations:
  - (i) Decision-making under certainty.
  - (ii) Decision-making under uncertainty.
  - (iii) Decision-making under risk.
  - (iv) Decision-making under conflict.

## NOTES

3. The conditional value of each event in the pay-off table is multiplied by its probability and the product is summed up. The resulting number is the EMV for the act.
4. The difference between the greater pay-off and the actual pay-off is known as opportunity loss.
5. The expected value with perfect information is the average (expected) return in the long run if we have perfect information before a decision is to be made.

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### 13.5 SUMMARY

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- Decision-making is an everyday process in life. It is the major role of a manager too. The decision taken by a manager has far reaching effect on the business. Right decisions will have salutary effect and the wrong one may prove to be disastrous.
- These days, in every organization whether large or small, the person at the top management has to take some decision, knowing that certain events beyond his control may occur to make him regret the decision. He is uncertain as to whether or not these unfortunate events will happen. In such situations, the best possible decision can be made by the use of statistical methods.
- In this case, the decision-maker knows with certainty the consequences of every alternative or decision choice. The decision-maker presumes that only one state of nature is relevant for his purpose.
- When the decision-maker faces multiple states of nature but he has no means to arrive at probability values to the likelihood of occurrence of these states of nature, the problem is a decision problem under uncertainty. Such situations arise when a new product is introduced in the market or a new plant is set up. In business, there are many problems of this nature.
- When probabilities can be assigned to the various states of nature, it is possible to calculate the statistical expectation of gain for each course of action. The conditional value of each event in the pay-off table is multiplied by its probability and the product is summed up.

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### 13.6 KEY WORDS

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- **Pay-off table:** A table that represents the economics of a problem, i.e., revenue and costs associated with any action with a particular outcome
- **Opportunity loss:** The loss incurred because of failure to take the best possible action
- **Maximax decision criterion:** An action chosen by the decision-maker that would result in the maximum possible pay-off



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## 13.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

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### Short Answer Questions

1. What is an event?
2. What is a pay-off table?
3. What is Laplace criterion?
4. Define the Hurwicz alpha criterion.
5. What is salvage value?

### Long-Answer Questions

1. What do you understand by ‘decision theory’?
2. Indicate the difference between decision under risk and decision under uncertainty?
3. Describe some methods which are useful for decision-making under uncertainty.
4. Explain the terms (i) Expected monetary value (ii) Expected value of perfect information.

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## 13.8 FURTHER READING

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- Taha, H.A. 2006. *Operations Research: An Introduction*, 8th edition. New Delhi: Prentice-Hall of India.
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### NOTES

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# UNIT 14 DECISION TREE ANALYSIS

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## NOTES

### Structure

- 14.0 Introduction
- 14.1 Objectives
- 14.2 Decision Making Environments
  - 14.2.1 Deterministic Decision Model (Decision-Making under Certainty)
  - 14.2.2 Probabilistic or Stochastic Decision Model (Decision-Making under Risk)
  - 14.2.3 Rules/Techniques for Decision-Making under Risk Situation
  - 14.2.4 Expected Profits with Perfect Knowledge (or Information) and the Expected Value of Perfect Information
  - 14.2.5 The Effect of Salvage Value
  - 14.2.6 Use of Marginal Analysis
- 14.3 Decision Tree Approach to Choose Optimal Course of Action
- 14.4 Concept of Posterior Probabilities
- 14.5 Answers to Check Your Progress Questions
- 14.6 Summary
- 14.7 Key Words
- 14.8 Self Assessment Questions and Exercises
- 14.9 Further Reading

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## 14.0 INTRODUCTION

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The Decision Tree Analysis is a schematic representation of several decisions followed by different chances of the occurrence. Simply, a tree-shaped graphical representation of decisions related to the investments and the chance points that help to investigate the possible outcomes is called as a decision tree analysis.

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## 14.1 OBJECTIVES

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After going through this unit, you will be able to:

- Describe the decision making environment
- Analyse the different models of decision making
- Explain the decision tree approach to choose optimal course of action
- Discuss the concept of Posterior Probabilities

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## 14.2 DECISION MAKING ENVIRONMENT

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There are various models used in decision-making. Some of the models are as follows.

### 14.2.1 Deterministic Decision Model (Decision-Making under Certainty)

Deterministic model is related to deterministic situation. Deterministic decision pay-offs are the simplest possible pay-offs. The objectives and strategies in this model have to be listed and then the pay-off for each strategy towards each objective is determined. For example, if there are two objectives  $O_1$  and  $O_2$ , the strategies to be selected are  $S_1$  and  $S_2$  and then the related pay-offs can be shown in matrix form as under:

<i>Objectives/Strategies</i>	$O_1$	$O_2$	<i>Total Pay-off <math>\Sigma a_{ij}</math></i>
$S_1$	$a_{11}$	$a_{12}$	$\Sigma a_{1j}$
$S_2$	$a_{21}$	$a_{22}$	$\Sigma a_{2j}$

Here  $a_{ij}$  ( $i = 1, 2; j = 1, 2$ ) refers to pay-offs of  $i$ th strategy towards  $j$ th objective. Total pay-off for strategy 1 is  $\Sigma a_{1j}$  (i.e.,  $a_{11}$  pay-off towards objective 1 and  $a_{12}$  pay-off towards objective 2) and for strategy 2 is  $\Sigma a_{2j}$ . The optimum strategy would be the one having the largest total pay-off (i.e., maximum of  $\Sigma a_{1j}$  and  $\Sigma a_{2j}$ ).

In general, with  $m$  objectives and  $n$  strategies the decision pay-off is as follows:

<i>Objectives/Strategies</i>	$O_1$	$O_2 \dots O_m$	<i>Total Pay-off</i>
$S_1$	$a_{11}$	$a_{12} \dots a_{1m}$	$\Sigma a_{1j}$
$S_2$	$a_{21}$	$a_{22} \dots a_{2m}$	$\Sigma a_{2j}$
.	.	.... .	.
.	.	.... .	.
.	.	.... .	.
$S_n$	$a_{n1}$	$a_{n2} \dots a_{nm}$	$\Sigma a_{nj}$

Here  $\Sigma a_{ij}$  again refers to pay-off of  $i$ th strategy towards  $j$ th objective. The optimum strategy in this case would be the one having the largest pay-off (i.e., maximum of  $\Sigma a_{1j}, \Sigma a_{2j}, \dots, \Sigma a_{nj}$ ).

The decision-making under certainty situation involves the following steps:

- (i) Determine the alternative courses of action.
- (ii) Calculate the pay-offs, one for each course of action.
- (iii) Select the alternative with largest profit or smallest cost either by the method of complete enumeration (if the number of alternatives is small) or with the aid of appropriate mathematical models.

### 14.2.2 Probabilistic or Stochastic Decision Model (Decision-Making under Risk)

Probabilistic model or what is known as the stochastic decision model is related to risk situation. Risk situation, as has already been explained, is one where there are many states of nature and the decision-maker knows the probability of occurrence of each such state. Decision pay-offs are not fixed but generally happen

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to be a random variable. Pay-offs are determined partly by chance and partly by the strategies adopted. Hence, in a probabilistic decision model, a decision is made in favour of that strategy which has the maximum expected pay-off.

Let us consider a simple example in which we have three objectives with three strategies. Objectives are denoted by  $O_1, O_2$  and  $O_3$  and strategies by  $S_1, S_2$  and  $S_3$ . The pay-off matrix can be stated as under:

<i>Objectives/Strategies</i>	$O_1$	$O_2$	$O_3$
$S_1$	$a_{11}$	$a_{12}$	$a_{13}$
$S_2$	$a_{21}$	$a_{22}$	$a_{23}$
$S_3$	$a_{31}$	$a_{32}$	$a_{33}$

The matrix of risk function (or probability) can similarly be denoted as under:

<i>Objectives/Strategies</i>	$O_1$	$O_2$	$O_3$
$S_1$	$p_{11}$	$p_{12}$	$p_{13}$
$S_2$	$p_{21}$	$p_{22}$	$p_{23}$
$S_3$	$p_{31}$	$p_{32}$	$p_{33}$

where  $ij$  refers to the probability of selecting  $i$ th strategy towards the achievement of  $j$ th objective. Also  $p_{ij}$  for all  $i, j$  or  $\sum p_y = 1$ .

After knowing the above stated two matrices, the next step is to calculate the expected pay-offs  $ij$  which can also be termed as Expected Monetary Value (or EMV).  $\sum E_{ij}$  is equal to the multiplication of decision pay-off elements to the corresponding probabilities. The expected pay-off matrix would be as follows:

<i>Objectives/Strategies</i>	$O_1$	$O_2$	$O_3$	<i>Total Expected Pay-off (or EMV)</i>
$S_1$	$E_{11}$	$E_{12}$	$E_{13}$	$\sum E_{1j}$
$S_2$	$E_{21}$	$E_{22}$	$E_{23}$	$\sum E_{2j}$
$S_3$	$E_{31}$	$E_{32}$	$E_{33}$	$\sum E_{3j}$

The best strategy in this case would be the one having the largest total expected pay-off or the EMV (i.e., maximum of  $\sum E_{1j}, \sum E_{2j}$  and  $\sum E_{3j}$ ). The similar treatment can be extended for  $n$  strategies and with  $m$  objectives.

**14.2.3 Rules/Techniques for Decision-Making under Risk Situation**

There are several rules and techniques for decision-making under risk situation. Important ones are:

1. Maximum likelihood rule.
2. Expected pay-off criterion:
  - (i) EMV criterion
  - (ii) EOL criterion
3. Decision trees.

4. Utility functions or the utility curves.
5. Bayesian decision rule (Posterior analysis)

We shall now explain all these decision rules and techniques one by one.

### 1. Maximum likelihood rule

Under this rule the decision-maker selects the most likely alternative. For example, if the probability distribution of demand is as follows:

<b>Demand (Units)</b>	<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>
<b>Probability</b>	0.1	0.1	0.4	0.05	0.05	0.2	0.1

As per the table, the most likely demand is a demand of 2 units and if one has to place the order, then he should place for 2 units as per the most likelihood rule. The disadvantage of this rule is that no consideration is given to less likely but more consequential results.

### 2. Expected Pay-off Criterion (EMV and EOL)

**Expected Monetary Value (or EMV) Criterion:** For a probabilistic decision model, the usual criterion is that of Expected Monetary Value. The decision-maker will have to adopt the following steps to work out the EMV:

- (i) The decision-maker should clearly state all possible actions that he thinks reasonable for the period (or periods) in question and also the possible outcomes of the actions.
- (ii) The decision-maker must then state the probability distribution concerning each possible action for which purpose he may use either a-priori or empirical methods of calculating probabilities. In simple words, the decision-maker should assign a probability weight to each of the possible actions or the states of nature.
- (iii) The decision-maker must finally use some yardstick (usually rupees) that measures the value of each outcome. Thus in other words, it means that the decision-maker should compute for each state of nature the consequences of the given act.
- (iv) He can then calculate the total expected pay-off (or expected monetary value) concerning each action and its outcome. This is done by summing up the product of the probability of each state of nature and the consequences of that act.
- (v) The action and outcome with the highest expected value should be finally selected.

In context of EMV, it should be kept in view that EMV technique is adequate only in those cases where the potential losses are not too great and the perspective

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profit range is narrow. But in problems which involve large potential losses, some other techniques of decision-making are generally adopted.

Let us illustrate all this by examples.

**Example 14.1:** Suppose a businessman wants to decide whether to stock commodity *X* or commodity *Y*. He can stock either but not both. If he stocks *X* and if it is a success, he feels that he can make ₹ 200 but if it is a failure he will lose ₹ 500. If he stocks *Y* and if it is a success he feels that he can make ₹ 400, but if it is a failure he would lose ₹ 300. The question is, which commodity *X* or *Y* should he stock? He has the following probability distribution in view:

<i>Probability</i>	<i>With Stock of</i>	<i>With Stock of Commodity X</i>
<b>Commodity Y</b>		
Success	0.80	0.60
Failure	0.20	0.40

**Solution:** We can write the pay-off matrix (in rupee terms) as follows for the given information:

<i>Objective/Strategy</i>	<i>Success</i>
<i>Failure</i>	
Commodity <i>X</i> +200 -500	
Commodity <i>Y</i> +400 -300	

The probability matrix being already given in the question, we can write the expected pay-off (or EMV) matrix as under:

<i>Objective/Strategy</i>	<i>Success</i>	<i>Failure</i>
<b>Total Expected Pay-off (or EMV)</b>		
Commodity <i>X</i> 160 - 100 = +60	(0.8) (200)	(0.2) (-500)
Commodity <i>Y</i> 240 - 120 = +120	(0.6) (400)	(0.4) (-300)

From the above matrix it is clear that EMV of stocking commodity *X* is (+) 60 rupees and that of stocking commodity *Y* is (+) 120 rupees. Clearly, the businessman should choose to stock commodity *Y*.

**Interpretation of EMV**

EMV should be correctly interpreted and understood by a decision-maker. In the given problem the expected monetary value is ₹ 120 which does not mean an assured profit of ₹ 120. It is just the expected value of making a profit. It simply means that if the businessman made this decision of stocking commodity *Y* several times, he would on an average make a profit of ₹ 120. But if he stocks commodity *Y*, say just once, he may even lose ₹ 300. The correct conclusion is that the chances of a greater profit are there with the stocking of commodity *Y*.

**Example 14.2:** Suppose a grocer is faced with a problem of how many cases of milk to stock to meet tomorrow's demand. All the cases of milk left at the end of the day are worthless. Each case of milk is sold for ₹ 8 and is purchased for ₹ 5. Hence, each case sold brings a profit of ₹ 3 but if it is not sold at the end of the day, then it must be discarded resulting in a loss of ₹ 5. The historical record of the number of cases of milk demanded is as follows:

<i>No. of Cases of Milk Demanded</i>	<i>No. of Times Demanded</i>	<i>Probability of Each Event</i>
0—12	0	0.00
13	5	0.05
14	10	0.10
15	20	0.20
16	30	0.30
17	25	0.25
18	10	0.10
Over 18	0	0.00
Total	100	1.00

What should be the optimal decision of the grocer concerning the number of cases of milk to stock? Assuming that the grocer has a perfect knowledge, evaluate what would be his expected profits?

**Solution:** The situation facing the grocer can be structured in the form of a matrix which shows conditional values (or conditional profits) as follows:

**Matrix of Conditional Values (or Profits)**

<i>Event: Demand or State of Nature</i>	<i>Possible Action Concerning Stock Policy</i>					
	<i>Stock</i>	<i>Stock</i>	<i>Stock</i>	<i>Stock</i>	<i>Stock</i>	<i>Stock</i>
13	39	34	29	24	19	14
14	39	42	37	32	27	22
15	39	42	45	40	35	30
16	39	42	45	48	43	38
17	39	42	45	48	51	46
18	39	42	45	48	51	54

Demand upto 12 cases and also demand over 18 cases have not been considered in the above matrix for their probabilities are zero on the basis of the past information.

It may be pointed out here that conditional value (or profit) means the actual profit which would result following a given action, conditional upon a given event occurring. Thus, in the given case, for example, if 16 cases of milk are ordered and 14 are sold, then the conditional profit would be ₹ 32 to be worked out as under:

$$\text{Profit per case of milk sold} = ₹ 3$$

$$\text{Profit of 14 cases sold} = ₹ 42$$

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2 cases remain unsold and are worthless resulting in a loss of ₹ 5 per case.

Total loss on 2 cases = ₹ 10

∴ Conditional profit = (₹ 42 – ₹ 10) = ₹ 32

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Conditional profit for all other possible actions can be worked out in a similar manner and a matrix of conditional profits (values) can be prepared.

The best action to be taken (or the optimal decision) is found by calculating the EMV for each stock action and then choosing the highest EMV. This can be done as shown here:

<i>Event: Demand</i>	<i>Probability of Each Event</i>	<i>Conditional Profit of Possible Action Stock 13</i>	<i>Expected Value of Possible Action Stock 13</i>
13	0.05	39	1.95
14	0.10	39	3.90
15	0.20	39	7.80
16	0.30	39	11.70
17	0.25	39	9.75
18	0.10	39	3.90

Expected monetary value = 39.00 (of stocking 13 cases of milk)

Similarly, the EMV for all other possible actions of stocking milk cases can be worked out which would be as follows:

<i>Possible Action</i>	<i>EMV</i>
Stock 14	$0.05(34)+0.10(42)+0.20(42)+0.30(42)+0.25(42)+0.10(42)=41.6$
Stock 15	$0.05(29)+0.10(37)+0.20(45)+0.30(45)+0.25(45)+0.10(45)=43.4$
Stock 16	$0.05(24)+0.10(32)+0.20(40)+0.30(48)+0.25(48)+0.10(48)=43.6$
Stock 17	$0.05(19)+0.10(27)+0.20(35)+0.30(43)+0.25(51)+0.10(51)=41.4$
Stock 18	$0.05(14)+0.10(22)+0.20(30)+0.30(38)+0.25(46)+0.10(54)=37.2$

The highest EMV is 43.6 rupees corresponding to action of stocking 16 cases of milk. Thus, the optimal course of action under the given condition of risk is to stock 16 cases of milk.

### 14.2.4 Expected Profits with Perfect Knowledge (or Information) and the Expected Value of Perfect Information

Perfect knowledge means that the decision-maker knows demand when he orders the goods. In such a situation, the following conditional values would be relevant:

<i>Event: Demand</i>	<i>Conditional Value</i>
13	With possible action: Stock 13 ₹ 39
14	With possible action: Stock 14 ₹ 42
15	With possible action: Stock 15 ₹ 45
16	With possible action: Stock 16 ₹ 48
17	With possible action: Stock 17 ₹ 51
18	With possible action: Stock 18 ₹ 54



Expected profits with perfect knowledge now can be worked out as under:

<i>Event:</i>	<i>Probability Demand</i>	<i>Conditional Value with Perfect Knowledge</i>	<i>Expected Value</i>
13	0.05	39	1.95
14	0.10	42	4.20
15	0.20	45	9.00
16	0.30	48	14.40
17	0.25	51	12.75
18	0.10	54	5.40

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- ∴ Expected profits with perfect knowledge = ₹ 47.70  
(or EMV with condition of certainty)
- ∴ EMV with condition of certainty is ₹ 47.70
- ∴ EMV with condition of risk is ₹ 43.60. Hence, the expected value of perfect information =  $(47.70 - 43.60) = ₹ 4.10$  being the value of transforming risk into certainty.

**EOL Criterion:** The grocer can also choose the best act in the given problem by minimizing expected opportunity loss. For this purpose a matrix showing conditional opportunity losses can be prepared as follows:

It may be pointed out here that conditional opportunity loss means the relative loss (i.e., the profit not earned) following a given action and conditional upon a given event occurring. Thus, in the given case, for example, if 16 cases of milk are ordered and 14 are sold, the conditional loss would be ₹ 10 (i.e., loss of ₹ 5 per case of unsold milk). If 13 cases are ordered and the demand happens to be of 18 cases, then the grocer would not be able to make a profit of ₹ 15 which he could have made had he ordered for 18 cases of milk. Thus, ₹ 15 is the conditional or opportunity loss in this concerning event for the grocer loses the opportunity to sell 5 additional cases of milk. Conditional opportunity losses for all other possible actions can be worked out in a similar manner and exhibited in the matrix of conditional opportunity losses as given below.

**Matrix of Conditional Opportunity Losses**

<i>Event: Demand or State of Nature</i>	<i>Possible Actions Concerning Stock Policy</i>					
	<i>Stock 13</i>	<i>Stock 14</i>	<i>Stock 15</i>	<i>Stock 16</i>	<i>Stock 17</i>	<i>Stock 18</i>
13	0	5	10	15	20	25
14	3	0	5	10	15	20
15	6	3	0	5	10	15
16	9	6	3	0	5	10
17	12	9	6	3	0	5
18	15	12	9	6	3	0

The best action to be taken (or the optimal decision) is found by calculating the EOL for each stock action and then choosing the smallest EOL. This can be done as follows:

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<i>Event: Demand</i>	<i>Probability of Each Event</i>	<i>Conditional Opportunity Loss of Possible Action</i>	<i>Expected Value</i>
13	0.05	0	0.00
14	0.10	3	0.30
15	0.20	6	1.20
16	0.30	9	2.70
17	0.25	12	3.00
18	0.10	15	1.50

Expected opportunity loss = 8.70 rupees of stocking 13 cases of milk.

Similarly, the EOL for all other possible actions of stocking milk cases can be worked out which would be as given here:

<i>Possible Action</i>	<i>EOL (Rupees)</i>
Stock 14	$0.05(5) + 0.10(0) + 0.20(3) + 0.30(6) + 0.25(9) + 0.10(12) = 6.1$
Stock 15	$0.05(10) + 0.10(5) + 0.20(0) + 0.30(3) + 0.25(6) + 0.10(9) = 4.3$
Stock 16	$0.05(15) + 0.10(10) + 0.20(5) + 0.30(0) + 0.25(3) + 0.10(6) = 4.1$
Stock 17	$0.05(20) + 0.10(15) + 0.20(10) + 0.30(5) + 0.25(0) + 0.10(3) = 6.3$
Stock 18	$0.05(25) + 0.10(20) + 0.20(15) + 0.30(10) + 0.25(5) + 0.10(0) = 10.5$

The smallest EOL is ₹ 4.1 corresponding to action of stocking 16 cases of milk which is the optimum action under given condition of risk. This answer is exactly the same as we had worked out with EMV.

#### 14.2.5 The Effect of Salvage Value

In the above illustration it was assumed that the unsold cases of milk left at the end of the day were completely worthless but in real life such unsold quantity may have some value known as the 'salvage value'. The effect of such salvage value is that it reduces the loss from overstocking. Suppose the salvage value of a case of milk remaining unsold is ₹ 2, then the overstock loss would be  $(₹ 5 - ₹ 2) = ₹ 3$  per case of milk. The best action would be worked out as per the procedure outlined above keeping in view the over stock loss after making adjustment concerning the salvage value.

#### 14.2.6 Use of Marginal Analysis

In many problems the procedure to find out the best action either through EMV or through EOL would be a tedious one because of the number of computations required. Marginal analysis provides the alternative to this but only in cases where the gains increase (or losses decrease) linearly. In our example, each additional case of milk sold brings a gain of ₹ 3 and each case of milk not sold causes a loss of ₹ 5. Thus, the given question satisfies the linearity assumption.

Marginal analysis starts considering that an additional unit bought will either be sold or it will not be sold. If  $p$  represent the probability of selling one additional unit, then  $(1 - p)$  must be the probability of not selling the additional unit, for the

sum of the probabilities of these two events must be one. If the additional unit purchased is sold, we shall realize an increase in our conditional profit and such an increase is known as the marginal profit ( $MP$ ). But if the additional unit purchased is not sold, then it will reduce our conditional profit and the amount of reduction is known as the marginal loss ( $ML$ ).

Thus the additional units should be stocked (purchased) as long as the expected marginal profit from stocking each of them is greater than the expected marginal loss from stocking each. The size of each order should be increased upto the point where  $MP$  and  $ML$  are equal. Thus, we should stock upto the point where expected marginal profit [ $p(MP)$ ] is greater than or equal to the expected marginal loss [ $(1-p)(ML)$ ]. In symbolic form we can state as follows:

Stock upto the point where,

$$p(MP) \geq (1-p)(ML)$$

or, 
$$p(MP) \geq ML - pML$$

or, 
$$p(MP) + p(ML) \geq ML$$

or, 
$$p \geq \frac{ML}{ML + MP}$$

**Example 14.3:** Solve the problem given in Example 9.18 by applying the marginal analysis.

**Solution:** An additional case of milk stocked (purchased) and sold increases the conditional profit by ₹ 3. Hence marginal profit or  $MP = ₹ 3$ .

An additional case of milk stocked but not sold causes reduction in the conditional profit by ₹ 5. Hence marginal loss or  $ML = ₹ 5$ .

Hence, stock upto the point where,

$$p \geq \frac{ML}{ML + MP}$$

or, 
$$p \geq \frac{5}{5 + 3}$$

or, 
$$p \geq 0.625$$

The given probability distribution concerning demand is as follows:

<i>Event: Demand</i>	<i>Probability</i>
13	0.05
14	0.10
15	0.20
16	0.30
17	0.25
18	0.10

The above table shows that the probability of selling 18 cases of milk is 0.10. The probability of selling 17 cases of milk or more is,

$$(0.25 + 0.10) = 0.35$$

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The cumulative probability of selling 16, 15, 14 and 13 cases of milk can be worked out in the same manner and can be put as under:

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<i>Demand (Sales) of</i>	<i>Cumulative Probability Cases of Milk</i>
13 or more	1.00
14 or more	0.95
15 or more	0.85
16 or more	0.65
17 or more	0.35
18 or more	0.10

Since the cumulative probability of selling 14 or more is 0.95 we should stock 14 cases of milk (because  $p$  is greater than 0.625). On similar reasoning we should clearly stock 15 cases of milk. We can as well stock 16 cases of milk (because  $p$  is greater than 0.625). But after this we find  $p$  only 0.35 for 17 or more and 0.10 for 18 or more which values are less than 0.625. Hence, the optimal action is to stock 16 cases of milk. This result is the same as we had worked out earlier through EMV and EOL techniques.

**The criteria for minimizing maximal regret and their applications have been discussed in Unit 3.**

### 14.3 DECISION TREE APPROACH TO CHOOSE OPTIMAL COURSE OF ACTION

Decision tree approach is a technique for making decision(s) especially in more complex risk situations. More complex decision problems can be solved conveniently using decision tree technique.

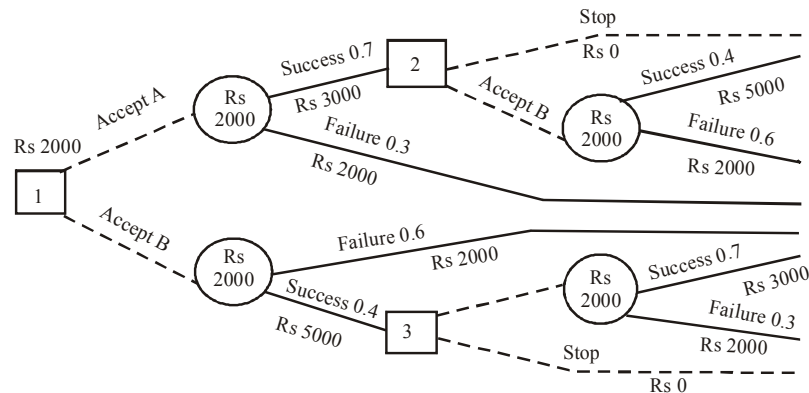
A decision tree (called tree as it looks like a tree) is a decision flow diagram that incorporates branches leading to alternatives one can select (the decision branches, often shown as dotted lines) among the usual branches leading to events that depend on probabilities (the probability branches, often shown as unbroken lines). In other words, a decision tree is a graphic way of showing the sequences of action-event combinations that are available to a decision-maker. Each sequence is shown by a distinct path through the tree. In the decision tree approach, we generally use the expectation principle, i.e., we choose the alternative that minimizes expected profit or the alternative that minimizes expected cost. The objective of using a decision tree is to decide which of the available alternative should in fact be selected.

The preparation of a decision tree can be better understood by a decision tree problem. Let us take some decision problems for this purpose.

**Example 14.4:** A businessman has to select two independent investments  $A$  and  $B$  available to him but he lacks to undertake the capital of both of them simultaneously. He can choose  $A$  first and then stop, or if  $A$  is successful then take  $B$ , or vice versa. The probability of success on  $A$  is 0.7, while for  $B$  it is 0.4. Both

investments require an initial capital outlay of ₹ 2000 and both return nothing if the venture is unsuccessful. Successful completion of *A* will return ₹ 3000 (over cost), successful completion of *B* will return ₹ 5000 (over cost). Draw the decision tree and determine the best strategy.

**Solution:** The decision tree of the problem can be drawn as follows:



The squares marked 1, 2 and 3 are the decision points and the circles represent event nodes. The expected pay-off values have been shown in the circles. The optimal decision is to accept *A* first and if successful, then accept *B* as the expected pay-off if this decision happens to be the maximum possible.

### Explanation

To construct the above decision tree, the technique consists of the forward pass, the backward pass and finally the reading of the tree.

### The Forward Pass

In this method, we first draw a square (the first decision point) on the left hand side of the tree. Alternatives available at this point (in our case *A* and *B*) are shown in dotted lines (the decision branches) leading to circles (the event nodes) from where the unbroken lines (the probability branches) are drawn (representing success and failures of *A* and also of *B* in our case). The probabilities and the related pay-offs are indicated on the probability branches. The probability branches representing success of *A* and *B* reach decision points 2 and 3, respectively. At the decision point 2, we may either stop or accept *B* and at decision point 3 we may either stop or accept *A*. The facts are shown as dotted lines and they lead us to new chance nodes from where again probability branches are drawn as required to complete the tree for the problem.

### The Backward Pass

The next step in a decision tree analysis is to make the backward pass. To do this, we write against every final event on the tree, the value of that event. In a backward pass, the following rules are to be complied with:

- (i) If the branches are probability branches, write the expectation in the relevant junction (or the node) of the branches.

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- (ii) If the branches are decision branches, select the branch with the highest expectation as the branch to be chosen and write this expectation alongside the junction of the branches (or the square). The selection may be indicated by ticking the decision branch.

By observing these rules, the relevant figures have been written in the circles and alongside the squares in the diagram of the tree drawn for the given problem.

**Reading the Completed Tree**

To read the completed tree we begin at the start and simply follow the ticked decision branches. In our example, we will say that the businessman should first accept investment *A* and if successful, then he should accept *B*.

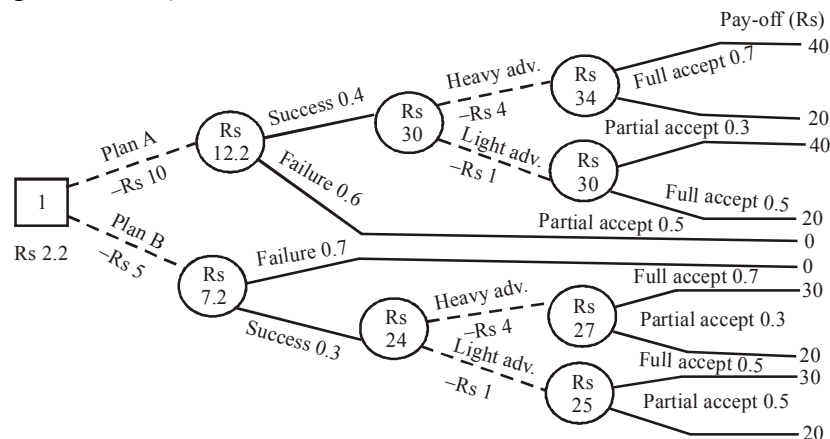
In context of a decision tree the following are important:

- (i) The overall expectation is given by the figure at the start.  
(In our example, it is ₹ 2000, the return on an average from investment.)
- (ii) Once a decision branch has been eliminated, all the subsequent parts of the tree become irrelevant. (In our case, the bottom half of the tree is irrelevant).

**Example 14.5:** Mr *X* of *ABC* Ltd. wants to introduce a new product in the market. He has a choice of two different research and development plans *A* and *B*. *A* costs ₹ 10 lakh and has 40 per cent chance of success, whereas *B* costs ₹ 5 lakh with 30 per cent chance of success. In the event of success, Mr *X* has to decide whether or not to advertise the product heavily or lightly. Heavy advertising will cost ₹ 4 lakh but gives a 0.9 probability of full acceptance and 0.3 probability of partial acceptance by the market. Light advertising will cost ₹ 1 lakh with a probability 0.5 of full acceptance. Full market acceptance of the product developed as per plan *A* would be worth ₹ 40 lakh and as per plan *B* would be worth ₹ 30 lakh. Partial acceptance in both the cases will be worth ₹ 20 lakh. Which plan should Mr *X* adopt and what sort of advertising should be done for marketing the product? Solve the problem with the help of a decision tree.

**Solution:** The decision tree of the problem can be drawn as follows:

(All figures in lakhs)



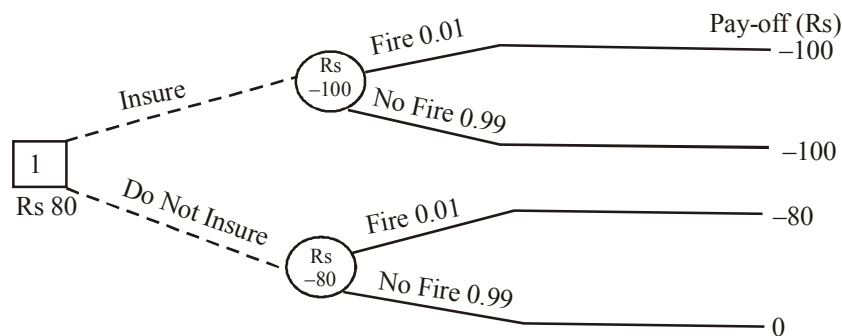
From the decision tree we find that Mr *X* should adopt Plan *B* and if successful should do light advertising for marketing the product. This decision will result in the maximum possible expected pay-off of ₹ 2.2 lakh.

**Example 14.6:** Prepare a decision tree for the information given in the following decision matrix relating to cost data:

<i>State of Nature</i>	<i>Fire</i> <i>0.01</i>	<i>No Fire</i> <i>0.99</i>
Alternatives		
Insure	₹ 100	₹ 100
Do Not Insure	₹ 80	₹ 0

Also, state what action should be taken to minimize cost?

**Solution:** The decision tree for the problem would be as follows:



As per the decision tree, the action should be not to insure as it would minimize the expected cost.

### Advantages of Decision Trees

Decision tree approach of decision-making under risk situations provides the following advantages:

1. It provides a graphic presentation of sequential decision processes.
2. It shows at a glance when decisions are expected to be made along with their possible consequences and the resultant pay-offs.
3. It simplifies the decision analysis as the results of the computations are shown on the tree itself.
4. It provides solution of complicated problems with relative ease.

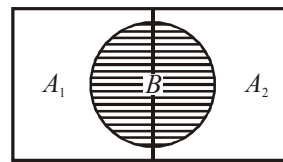
## 14.4 CONCEPT OF POSTERIOR PROBABILITIES

Having discussed joint and conditional probabilities, one should try to investigate how probabilities are revised to take account of new information. Bayes' theorem (named after Thomas Bayes' an English Philosopher) published in 1763 in a short paper constitutes a unique method for calculating revised probabilities. In other

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words, this theory concerns itself to the question of determining the probability of some even  $E_i$  given that another event  $A$  has been (or will be) observed, i.e., the theory determines the value of  $P(E_i/A)$ . The event  $A$  is generally thought of a sample information. As such Bayes' rule is concerned with determining the probability of an even. For example, consider a machine which is not working correctly. Given are certain sample information, say the probability of defective article is 4 per cent on the basis of sample study. Bayes' theorem has many important applications evaluating the worth of additional information specially in context of decision analysis. To illustrate this, let  $A_1$  and  $A_2$  be the set of events which are mutually exclusive and exhaustive, so that  $P(A_1) + P(A_2) = 1$  and  $B$  be a simple event which intersects each of the  $A$  events as shown in the diagram given below.



In the above diagram the part of  $B$  which is within  $A_1$  represents the area ' $A_1$  and  $B$ ' and the part of  $B$  within  $A_2$  represents the area ' $A_2$  and  $B$ '. This being so the probability of event  $A_1$  given event  $B$  is.

$$P(A_1 | B) = \frac{P(A_1 \text{ and } B)}{P(B)}$$

and the probability of event  $A_2$  given event  $B$  is,

$$P(A_2 | B) = \frac{P(A_2 \text{ and } B)}{P(B)}$$

Where,  $P(B) = P(A_1 \text{ and } B) + P(A_2 \text{ and } B)$

$$P(A_1 \text{ and } B) = P(A_1) \times P(B | A_1)$$

and,  $P(A_2 \text{ and } B) = P(A_2) \times P(B | A_2)$

In general if  $A_1, A_2, A_3, \dots, A_n$  be the set of mutually exclusive and exhaustive events, then the above expressions may be stated as follows:

$$P(A_i | B) = \frac{P(A_i \text{ and } B)}{P(B)}$$

Where,  $P(B) = P(A_1 \text{ and } B) + P(A_2 \text{ and } B) + \dots + P(A_n \text{ and } B)$

$$P(A_i \text{ and } B) = P(A_i) \times P(B | A_i)$$

A probability that undergoes revision through above stated Bayes' rule in the light of sample information is called a *posterior probability*. Posterior probabilities are always conditional probabilities. We shall explain this by an example given below:



**Example 14.7:** It is not known whether a coin is fair or unfair. If the coin is fair, the probability of tail is 0.5 but if the coin is unfair the probability of a tail is 0.10. A-priori or unconditional probability given of a fair coin is 0.80 and that of unfair coin is 0.20. The coin is tossed once and tail is the result. (i) What is the probability that the coin is fair? (ii) What is the probability that the coin is unfair?

**Solution:** Let the event 'fair coin' be designated by  $A_1$  and the event 'unfair coin' by  $A_2$ . Then the given information can be put as under:

$$\left. \begin{array}{l} P(A_1) = 0.80 \\ P(A_2) = 0.20 \end{array} \right\} \text{A-priori (or unconditional) probabilities}$$

$$\left. \begin{array}{l} P(\text{tail}/A_1) = 0.5 \\ P(\text{tail}/A_2) = 0.1 \end{array} \right\} \text{Conditional probabilities}$$

$$\left. \begin{array}{l} P(\text{tail and } A_1) = P(A_1) \times P(\text{tail}/A_1) \\ \quad = (0.8) (0.5) \\ \quad = 0.40 \\ P(\text{tail and } A_2) = P(A_2) \times P(\text{tail}/A_2) \\ \quad = (0.2) (0.1) \\ \quad = 0.02 \end{array} \right\} \text{Joint probabilities}$$

A tail can occur in combination with 'fair coin' or in combination with 'unfair coin'. The probability of the former is 0.40 and of the latter is 0.02. The sum of the probabilities would result in the unconditional probability of a tail on the first toss, i.e.,

$$P(\text{tail}) = 0.40 + 0.02 = 0.42$$

Thus if a tail occurs and if it is not known whether the coin tossed once is fair coin or unfair coin, then the probability of its being a fair coin is:

$$P(A_1 \text{ tail}) = \frac{P(\text{tail and } A_1)}{P(\text{tail})} = \frac{0.40}{0.42} = 0.95$$

This is the posterior (or revised) probability of a fair coin (or  $A_1$ ) given that tail is the result in the first toss of a coin obtained through Bayes' rule explained above.

We can similarly calculate the posterior probability of an unfair coin (or  $A_2$ ) given that tail is the result in the first toss and it can be shown as follows:

$$P(A_2 \text{ tail}) = \frac{P(\text{tail and } A_2)}{P(\text{tail})} = \frac{0.02}{0.42} = 0.05$$

Thus the revised probabilities after one toss when the toss results in tail are 0.95 of a fair coin and 0.05 of an unfair coin (initially they were 0.80 and 0.20, respectively).

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**Check Your Progress**

1. What is the decision tree approach?
2. What are the rules which need to be complied with during a backward pass?
3. What are the advantages of a decision tree?

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**14.5 ANSWERS TO ‘CHECK YOUR PROGRESS’**

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1. Decision tree approach is a technique for making decision(s) especially in more complex risk situations. More complex decision problems can be solved conveniently using decision tree technique.

A decision tree (called tree as it looks like a tree) is a decision flow diagram that incorporates branches leading to alternatives one can select (the decision branches, often shown as dotted lines) among the usual branches leading to events that depend on probabilities (the probability branches, often shown as unbroken lines). In other words, a decision tree is a graphic way of showing the sequences of action-event combinations that are available to a decision-maker.

2. In a backward pass, the following rules are to be complied with:
  - (i) If the branches are probability branches, write the expectation in the relevant junction (or the node) of the branches.
  - (ii) If the branches are decision branches, select the branch with the highest expectation as the branch to be chosen and write this expectation alongside the junction of the branches (or the square). The selection may be indicated by ticking the decision branch.
3. The advantages of decision trees are:
  - (i) It provides a graphic presentation of sequential decision processes.
  - (ii) It shows at a glance when decisions are expected to be made along with their possible consequences and the resultant pay-offs.
  - (iii) It simplifies the decision analysis as the results of the computations are shown on the tree itself.
  - (iv) It provides solution of complicated problems with relative ease.

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**14.6 SUMMARY**

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- Deterministic model is related to deterministic situation. Deterministic decision payoffs are the simplest possible pay-offs. The objectives and strategies in this model have to be listed and then the pay-off for each strategy towards each objective is determined.

- Probabilistic model or what is known as the stochastic decision model is related to risk situation. Risk situation, as has already been explained, is one where there are many states of nature and the decision-maker knows the probability of occurrence of each such state.
- In many problems the procedure to find out the best action either through EMV or through EOL would be a tedious one because of the number of computations required. Marginal analysis provides the alternative to this but only in cases where the gains increase (or losses decrease) linearly.
- Decision tree approach is a technique for making decision(s) especially in more complex risk situations. More complex decision problems can be solved conveniently using decision tree technique.
- A decision tree (called tree as it looks like a tree) is a decision flow diagram that incorporates branches leading to alternatives one can select (the decision branches, often shown as dotted lines) among the usual branches leading to events that depends on probabilities (the probability branches, often shown as unbroken lines).
- In other words, a decision tree is a graphic way of showing the sequences of action-event combinations that are available to a decision-maker. Each sequence is shown by a distinct path through the tree.

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### 14.7 KEY WORDS

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- **Deterministic model:** Deterministic model is related to deterministic situation. Deterministic decision payoffs are the simplest possible pay-offs. The objectives and strategies in this model have to be listed and then the pay-off for each strategy towards each objective is determined.
- **Decision tree approach:** It is a technique for making decision(s) especially in more complex risk situations. More complex decision problems can be solved conveniently using decision tree technique.

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### 14.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

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#### Short Answer Questions

1. What is salvage value?
2. What is marginal analysis? Give a brief account.
3. Define decision tree.

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## Long Answer Questions

1. What are EMV and EOL criteria?
2. Distinguish pay-off table and regret table? Explain with the help of examples.
3. Explain decision trees for sequential decisions. Why are they used?
4. What is the significance of action space? Explain with the help of an example.
5. Explain (i) Maximax (ii) Minimax, and (iii) Maximin decision criteria.
6. From the pay-off table shown below, decide which is the optimal act.

<i>Events</i>	<i>Strategies</i>		
	$A_1$	$A_2$	$A_3$
$S_1$	20	40	60
$S_2$	15	-10	-15
$S_3$	35	25	-20
	$P(S_1) = 0.4$	$P(S_2) = 0.5$	$P(S_3) = 0.1$

EMV for  $A_1 = 19$ , for  $A_2 = 13.5$  and for  $A_3 = 13.5$ .

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## 14.9 FURTHER READING

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